elevation of 9,600 feet. Dinner was served here, and then the guests repaired to a camp-fire while tables were cleared and benches were rearranged for the more formal part of the evening. For this feature Professor Kempner first called on Dean Lester who welcomed the visitors on behalf of the University. Professor Hedrick then spoke as president of the Society, expressing our pleasure and the value of our meetings in this region, and emphasizing the wide-spread and outstanding character of the Society's work. Professor Cairns, representing the Association, told of its present activities, mentioning particularly the thought and planning being devoted to the Carus Monographs, the forthcoming Chace publication of the Ahmes Papyrus and the contemplated reestablishment of the Bibliotheca Mathematica. More definite announcements will be made within a few weeks concerning the Association's publications. Secretary Cairns proposed a vote expressing the appreciation of the visitors to the University authorities and mathematics faculty for their genial hospitality, to Professor Sisam and the program committee for the formation of a strong program, and to Professor Hutchinson and the other members of the recreations committee for their extensive plans for the comfort of the visitors and for the various outings of the week. This was a noteworthy feature of the Boulder meetings, for the recreation facilities of the university, so ably directed by Professor Hutchinson as a valuable adjunct of the summer quarter, were carried over for an extra week solely for the entertainment of the mathematicians. The motion of appreciation was carried heartily by a rising vote. Professor Emeritus DeLong gave an interesting series of reminiscences of the earlier years of the University and of the department of mathematics. Professor Hutchinson as the last speaker described more in detail the recreations department, its methods, purposes and ideals.

Wednesday was an open day and was devoted to an excursion into the mountains by university and private automobiles. The drive was made by the beautiful South St. Vrain Canyon and Estes Park village to a point near Horseshoe Falls, where lunch was served in the open by the university. In the afternoon the drive was continued from an elevation of about 8,500 feet to the summit of Fall River Pass, an elevation of 11,797 feet, well above the timberline. Clouds sweeping over the pass, a rainstorm a mile or two to the south, and numerous snowball fights made this trip noteworthy to many of the group not used to the mountain experiences. The party returned to Boulder by way of North St. Vrain Canyon in ample time for dinner. Other shorter trips were made by smaller groups during the week. Many planned their vacations so as to make a stay in the mountains for two or three weeks preceding or following the meetings.

The American Mathematical Society held its thirty-fifth summer meeting and thirteenth colloquium from Tuesday to Friday, with lectures by Professor R. L. Moore of the University of Texas on "Point set theory." More than ninety were present at the colloquium lectures. Sessions for the reading of papers were held on Tuesday afternoon, Thursday morning and afternoon and Friday morning.

The Mathematical Association held sessions on Monday afternoon and Tuesday morning, Professor Rietz presiding at the first session and Professor Bussey at the second session. The program was arranged by a committee consisting of Professors C. H. Sisam (chairman), E. W. Chittenden, E. E. DeCou, T. M. Putnam and S. W. Reaves. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

First Session of the Association

(1) "The undergraduate mathematical curriculum in a liberal arts college," by Professor F. L. Griffin, Reed College.

(2) "Differential equations as a foundation for electrical circuit theory," by Dr. T. C. Fry, Bell Telephone Laboratories, New York City.

(3) "Preliminary tests in mathematics for college freshmen," by Professor W. L. Hart, University of Minnesota.

1. The paper of Professor Griffin will appear in an early issue of the Monthly.

In comment on Professor Griffin's paper, Professor Hedrick stated that a considerable variation from the suggested scheme must be made to accord with the taste of the instructor and the wishes or the abilities of the students. He would like some one to undertake a condensed course covering modern geometry and descriptive geometry and perhaps including projective geometry; there is much material of a general nature in these fields that should be known to prospective teachers. We need also to make the values of mathematics evident in quarters where this appeal is not now felt; we shall convince the world of the importance of mathematics by actually making it useful in other fields.

2. The foundation for the widespread use of complex quantities in dealing with electrical circuit theory lies in certain simple properties of differential equations, and can be explained much more effectively in an elementary course in that subject than in one devoted to the more technical aspects of engineering. Because of this, Dr. Fry urged that the laying of such a foundation be made a part of courses on differential equations when a part of the class is composed of technical students and that the importance of the subject to these students be emphasized whenever possible.

The paper will appear in a later issue of the Monthly.

3. Professor Hart presented the results of a statistical investigation concerning 600 freshmen in classes in mathematics in the College of Arts at the University of Minnesota in the year 1926–27. One half of this group had had two units of mathematics in high school and the remainder had had two and one-half units. The data for the investigation consisted of the following items: (1) the performance of the students on taking the two examinations in mathematics of the Iowa series of placement tests, before the opening of college; (2) the subsequent performance of the students in their first course in college mathematics; (3) the scores obtained by some of the students on the general college ability rating scale used at the University of Minnesota. On the basis of the

results presented, conclusions were drawn concerning the preliminary and the placement plan which might be adopted for a non-homogeneous group of freshmen such as had been considered.

In comment upon the coefficient of correlation .6 Professor Gibson said that his experience is that in general a coefficient as large as this is of some significance, but that in this case it is probably not high enough to be significant, that in his judgment the scoring of college ability is defective. The availability of a correlation coefficient between the grades of the first and second examinations and the grade in the course is most to be desired except for the administrative difficulty of shifting pupils from one section to another better suited to their ability. We must usually fall back on placement tests such as are given at the various universities.

SECOND SESSION OF THE ASSOCIATION

(4) "Some aspects of ordinary differential equations," retiring presidential address, by Professor W. B. FORD, University of Michigan.

(5) "Factorization of numbers," by Professor D. N. LEHMER, University of California.

(6) "The application of groups to geometry," by Professor R. M. WINGER, University of Washington.

4. Professor Ford outlined at some length the Fuchs theory of ordinary differential equations and contrasted with this a general theory as developed by Dini. This theory is allied to the theory of integral equations and covers solutions not comprised by the old theory, besides formulating the theory for the solutions in a much more general form. It is to be hoped that Professor Ford will make this presentation available for the members of the Association, particularly since the integral equation methods and related theory play so prominent a part at present in the study of wave mechanics.

5. Professor Lehmer described a set of factor stencils which is being prepared by the Carnegie Institution of Washington, for the purpose of facilitating the finding of factors of numbers as high as two billion and a half. The theory on which the process rests, viz., certain theorems on quadratic residues, was also presented and illustrated. A fuller notice will appear soon in the Bulletin of the American Mathematical Society.

This supplements the well-known list of primes published by the Carnegie Institution in 1914 and other work done by Professor Lehmer and his son, D. H. Lehmer. See this Monthly for March 1928, pp. 114–121.

6. This paper limits itself to applications of finite groups in the binary and ternary domains. First symmetry is shown to be a metric aspect of group theory and the maximum symmetry of algebraic curves is considered. The geometry of configurations is also closely related to that of collineation groups. Every group has an invariant configuration and conversely certain configurations completely define allied finite groups. The two major problems in the theory of self-projective curves are mentioned: (1) the determination of the

complete system of invariant curves of a given group; (2) the determination of all varieties of groups that may leave a curve of given order invariant. Finally the special problem of self-projective rational curves is considered. In S_{n-1} the group on the parameter is one of the five species of binary (regular body) groups, while the group on the points is an n-ary group isomorphic with the binary group. The interplay between the geometry of the two associated groups is one of the interesting phases of the problem for rational curves.

This paper will appear in an early issue of the Monthly.

Meeting of the Board of Trustees

Eight trustees were present at the meetings on Monday evening and Tuesday noon.

The following twenty-six persons were elected to membership on applications duly certified:

To Individual Membership

J. B. Adkins, Ph.B. (Chicago). Teacher, Culver Military Acad., Culver, Ind.

E. F. Allen, B.S. (Acad. of the New Church). Instr., Academy of the New Church, Bryn Athyn, Pa.

Brother Aurelius, A.M. (Catholic Univ. of Amer.). Teacher, St. Joseph's College High School, Bardstown, Ky.

Jack Britton, A.B. (Clark). Inst., Univ. of Colorado, Boulder, Colo.

Alice Bromwell, A.M. (Nebraska). Instr., Monticello Seminary, Godfrey, Ill.

June F. Constantine, B.S. (Minnesota). Research Asst., Coll. of Educ., Univ. of Minnesota, Minneapolis, Minn.

W. J. Ettinger, B.S. in M.E. (Lewis Inst.). Research Engr., Edison Electric Appliance Co., Chicago, Ill.

Mary Ewin, A.B. (George Washington). Grad. Student, George Washington Univ., Washington, D. C.

Sidney Hacker, A.B. (Colorado). Part-time Instr., Univ. of Colorado, Boulder, Colo.

David Kevles, A.B. (Pennsylvania). 862 N. Marshall St., Philadelphia, Pa.

W. W. McCormick, B.S. (Geneva). Instr., Math. and Physics, Geneva Coll., Beaver Falls, Pa.

Charlotte L. McFall, A.M. (Chicago). Prof., West Virginia State College, Institute, West Va.

R. E. McPherson, M.S. (Chicago). Teacher, Glenn High School, Terre Haute, Ind.

G. E. March, B.S. (S. Dak. State School of Mines). Asst. Prof., South Dak. State School of Mines, Rapid City, S. Dak.

Florentina Mathias, A.M. (Ohio State). Teacher, Chillicothe High School, Chillicothe, Ohio.

Sigurd Mundhjeld, A.B. (Concordia, Moorhead, Minn.). Instr., Waldorf Coll., Forest City, Iowa.

H. A. Perkins, A.B. (Colby). Chairman Math. Group, Hampton Inst., Hampton Institute, Va.

E. W. Ploenges, A.M. (Michigan). Prof., Kansas Wesleyan Univ., Salina, Kans.

M. F. Rosskopf, A.B. (Minnesota). Teaching Asst., Univ. of Minnesota, Minneapolis, Minn.

E. A. Saibel, Ph.D. (Mass. Inst. of Tech.). Instr., Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.

Samuel Silberfarb, Ph.D. (Chicago). Asst. Prof., Univ. of Akron, Akron, Ohio.

Martha L. Smith, A.B. in Educ. (Virginia Union Univ.). Asst. Prof., Virginia Union Univ., Richmond, Va.

Marguerite Edna Stagner, B.S. (Iowa State). Teacher, High School, Glenham, S. Dak.

T. R. C. Wilson, C.E. (Purdue). Senior Engineer, Forest Products Lab., Madison, Wis.

Patrick Youtz, M.S. (Chicago). Instr., Bucknell Univ., Lewisburg, Pa.

Sister Yvonne, A.M. (Minnesota). St. Joseph's Acad., St. Paul, Minn.

The trustees gave further consideration to various routine matters of Association business and to the possibility of an improvement in the method of nominating candidates for offices in the Association.

W. D. Cairns, *Secretary-Treasurer*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-fifth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the George Washington University, Washington, D. C., on Saturday, May 4, 1929. Sessions were held in the morning and in the afternoon; Professor C. C. Bramble, Chairman of the Section, presided at both sessions.

Forty-four persons attended the meeting, including the following thirty-three members of the Association: O. S. Adams, W. J. Berry, L. M. Blumenthal, C. C. Bramble, Paul Capron, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. T. Erwin, P. J. Federico, Michael Goldberg, W. M. Hamilton, F. E. Johnston, L. M. Kells, W. D. Lambert, A. E. Landry, C. L. Leiper, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, John Tyler, C. E. Van Orstrand, W. J. Wallis, Paul Wernicke, C. H. Wheeler, 3d, E. W. Woolard, Oscar Zariski.

During the intermission between the morning and the afternoon sessions, those attending the meeting were entertained at luncheon by the Washington members. Preceding the reading of papers at the afternoon session, a brief business meeting was held, at which a vote of thanks was passed in appreciation of the hospitality of the local members and of the provisions made by the University for holding the meeting. The following officers were elected: *Chairman*, Professor W. F. Shenton, American University; *Secretary*, Edgar W. Woolard, George Washington University; *Members of the Executive Committee*, Professor H. M. Robert, Jr., U. S. Naval Academy, and Professor Florence P. Lewis, Goucher College. Professor Gwinner offered several suggestions as to policy concerning the conduct of the Section, for the consideration of the members and the new Executive Committee.

The following six papers were presented:

1. "Vector operations in projective geometry," by Professor Tobias Dantzig, University of Maryland.

2. "Some industrial engineering curves," by Professor Harry Gwinner, University of Maryland.

3. "The invalidity of a certain method of computing a probable error," by Professor J. B. Scarborough, U. S. Naval Academy.

4. "Some interesting formulae for the constant pi," by J. A. Duerksen, U. S. Coast and Geodetic Survey.

5. "On Einstein's new theory," by Professor F. D. Murnaghan, Johns Hopkins University.

6. "Comment on the problem of three listening posts," by Professor Paul Capron, U. S. Naval Academy.

Abstracts of some of these papers follow:

1. It is well known that the relations of projective geometry can be made

independent of all considerations of Cartesian metrics; it is less known that these relations are such as to permit the application, practically without restrictions, of the principles of vector algebra. This paper sought to show that vector analysis, so successfully used in differential geometry, can render just as signal a service in projective geometry. Examples were given to illustrate how the classical identities of vector algebra become theorems of projective geometry, and how a great number of geometrical constructions can be directly and intrinsically described by this method.

5. The essential features of the older generalized Theory of Relativity were described, and a brief exposition of the new form and the respects in which it differs from the preceding, were given.

6. A comment on the problem of three fixed stations in a straight line, at each of which a record is made of the time at which the sound of the discharge of a distant gun arrives: This problem is given in several texts as a problem in the intersection of two hyperbolas of known transverse axes with one of their three collinear foci in common. It was shown that consideration of the third hyperbola makes a simple solution possible that gives a brief computation for the desired position with no danger of mistaking an intersection of but two hyperbolas for the true position. The formulas obtained were seen to be readily usable for any order of succession of the arrival of the sound at the different posts, and to give ready means of comparing the accuracy of a plot of the actual curves with the plot of the asymptotes. It was also shown that there is always one false position; and criteria were given for the occurrence of two or of three such. The formula is useful in devising problems of any desired character; extension of the use of the formula to analogous problems concerning ellipses was made evident.

EDGAR W. WOOLARD, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association was held at Rice Institute, Houston, Texas, on Saturday, January 26, 1929. Professor G. T. Whyburn presided. The attendance was forty-three, including the following twenty members of the Association: J. H. Binney, L. W. Blau, A. A. Blumberg, H. E. Bray, Alice C. Dean, J. L. Dorroh, H. J. Ettlinger, G. C. Evans, L. R. Ford, E. Garza, H. Halperin, E. O. Lovett, E. R. C. Miles, W. L. Porter, W. A. Rees, W. T. Reid, J. H. Roberts, W. G. Smiley, Jr., P. H. Underwood, G. T. Whyburn.

The following members of the Association functioned as the local committee on arrangements: L. R. Ford, chairman, H. E. Bray and Alice Dean. The program committee consisted of G. C. Evans, chairman, G. T. Whyburn, and H. J. Ettlinger. The following was the program of the day:

1. "Generalized Vandermonde determinants and their applications to sym-

metric functions," E. R. Heineman, Texas Technological College (by invitation).

2. "A generalization of Hadamard's theorem on the absolute value of a determinant," E. F. Bechenbach, Rice Institute (by invitation).

3. "A fundamental continuity theorem in algebra," by W. T. Reid, University of Texas.

4. "Some properties of mortality curves," by L. R. Ford, Rice Institute.

5. "Some topics relating to the foundations of geometry," by J. L. Dorroh, University of Texas.

6. "Rates of foreign exchange," by G. C. Evans, Rice Institute.

7. "The polar form of a second order linear differential system," by H. J. Ettlinger, University of Texas.

8. "A report of the committee on requirements for teachers of mathematics," by Professor F. W. Sparks, Texas Technological College.

9. "Results of tests in mathematics given to freshman physics students," by L. W. Blau, University of Texas.

10. "Query: What should be the contents of the freshman mathematics course?," by C. R. Sherer, Texas Christian University. Discussion.

Abstracts of these papers follow:

1. A generalized Vandermonde determinant is obtained from the ordinary Vandermonde determinant by permitting the indices to take any set of values. The Vandermonde matrix is defined to be the Vandermonde determinant with the nth powers of its variables added as an extra row. By successively blocking out each of the first n rows of this matrix, we obtain n determinants which can be called secondary Vandermonde determinants. The ordinary Vandermonde determinant, which we get by omitting the last row of this matrix, will be called the principal Vandermonde determinant. It can be shown that every generalized Vandermonde determinant is expressible as a determinant-function of the principal and secondary Vandermonde determinants. These results, in conjunction with a theorem of Muir's, give a method for expressing any integral, rational, symmetric function in terms of elementary symmetric functions.

2. For definite Hermitian determinants Mr. Bechenbach extends the result of Frisch that $|\Delta| \leqq M^n$ or $|\Delta| \leqq \pi_{i=1}^{n} |\alpha_{ii}|$.

3. Mr. Reid discussed the theorem that the roots of an algebraic function are continuous functions of the coefficients. He pointed out that one must give an interpretation of the definition of continuity in this case.

Weber in his "Lehrbuch der Algebra" states the following theorem: "The roots of an algebraic equation are continuous functions of the coefficients." If the equation cannot be solved algebraically, the statement that each root of the equation is a continuous function of the coefficients has little meaning unless a method is given for discriminating between the roots of the equation. This question is considered and also, since the proof given by Weber is erroneous, a proof of the following theorem is given:

If $P_m(z) \equiv z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ is a polynomial which has roots

$r_1, r_2, \cdots, r_k$ with corresponding multiplicities $\alpha_1, \alpha_2, \cdots \alpha_n$, $\alpha_1+\alpha_2+\cdots+\alpha_n = m$, then for every $\epsilon>0$, there exists a $\delta_\epsilon>0$ such that if $\overline{P}_m(z) \equiv z^m+\bar{a}_{m-1}z^{m-1} + \cdots +\bar{a}_1 z+a_0$ is a polynomial such that $\sum_{i=0}^{m-1} |a_i-\bar{a}_i| < \delta_\epsilon$, then corresponding to each root r_i of $P_m(z)$ there exist α_i roots $\beta_1^{(i)}, \beta_2^{(i)}, \cdots, \beta_{\alpha_i}^{(i)}$ of $\overline{P}_m(z)$ such that $|r_j-\beta_j^{(i)}| < \epsilon$, $(j=1, 2, \cdots \alpha_i)$.

4. Professor Ford presented some elementary properties of mortality curves which seem to have escaped notice hitherto:

(1) At its maximum, the expectation of life is the reciprocal of the force of mortality.

(2) At its maximum, the life annuity is the reciprocal of the sum of the force of mortality and the force of interest.

(3) The most probable or least probable moment of death occurs when the derivative of the force of mortality equals the square of the force of mortality.

(4) A sum to be paid at death within a short interval of constant length is most or least expensive when the logarithmic derivative of the force of mortality equals the sum of the force of interest and the force of mortality.

5. In his paper, "Sets of metrical hypotheses for geometry" (Transactions of the American Mathematical Society, vol. 9 (1908), pp. 487–512), R. L. Moore has discussed the relationship between certain sets of his axioms and the groups I–IV used in Hilbert's *Grundlagen der Geometrie*. It is indicated, in the present discussion, how some recent proofs show that the groups of I–IV of Hilbert's axioms are satisfied in a space satisfying the sets *O* and *C* of Moore's axioms if his definition for the congruence of angles is used.

6. Professor Evans replaces the algebraic system of Cournot[1] governing rates of foreign exchange by corresponding differential equations.

7. Professor Ettlinger explained the general linear homogeneous system of a second order in the polar form and showed that when the rectangular coordinates are replaced by amplitude and angle functions many of the results of Sturm can be obtained directly from the properties of these functions.

8. During the afternoon session the secretary read a report of a committee of which F. W. Sparks is chairman, appointed at the 1928 meeting for the purpose of considering ways and means of improving the teaching of mathematics in Texas high schools. The committee presented the following resolutions:

(1) That prospective teachers of high school mathematics be urged to take a minimum of three years of college mathematics.

(2) That the State Board of Education be requested to place one-half year of algebra in the curriculum of the fourth year of the high school.

(3) That the general aim of the reorganization of high school mathematics courses be to conform with the recommendations of the College Entrance Examination Board and of the National Committee on Mathematical Requirements.

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[1] Cournot, *The Mathematical Principles of Wealth.*

(4) That the teachers colleges be requested to provide not only for better mathematical equipment of prospective teachers of mathematics but also for competent instruction in methods of teaching mathematics.

Professors Evans, Underwood, and Ettlinger commented at length on these resolutions. By a vote of the Section, Resolution (2) was adopted unanimously and the committee was requested to continue its study of the work in hand.

9. Mr. L. W. Blau, instructor in Physics in the University of Texas, presented some tests in arithmetic and elementary plane geometry given to freshman students at the University of Texas.

10. Prof. C. R. Sherer, head of the mathematics department at Texas Christian University, Fort Worth, presented the query, "What should be the contents of the freshman mathematics course?" This question was discussed from many angles by Professor Bray of Rice Institute, by Professor Ettlinger of the University of Texas, and by Professor Halperin of Texas A. & M. College.

Those in attendance were at Cohen House for lunch as guests of Rice Institute. The section voted a resolution of thanks to the members of Rice Institute for their splendid hospitality. Dr. G. T. Whyburn was re-elected chairman for a period of two years.

H. J. ETTLINGER, *Secretary*

ON THE DENSITY OF AN OBLATE SPHEROIDAL PLANET AND THE MOTION OF A SATELLITE

By LOUIS ALLEN HOPKINS, University of Michigan.

In his dissertation on *Periodic orbits about an oblate spheroid*, W. D. MacMillan,[1] proceeding by methods due to Poincaré, secured the following simple formula for the advance of the line of apsides of the planet per revolution, viz.,

$$\alpha = 360^\circ\left[\frac{3b^2}{10a^2}\mu^2(1 + e^2 + \cdots)\right]. \tag{1}$$

In this expression b is the polar radius of the planet, μ the eccentricity of all meridian cross sections, a the mean distance of the satellite, and e the eccentricity of its orbit. In making application of his theory to the motion of the fifth satellite of Jupiter, MacMillan found a wide difference between the computed and the observed values of α. However he supposed that the density of Jupiter was uniform. The object of this paper is to discuss consequences of distributions of density which are homogeneous in concentric oblate spheroidal shells and which are biquadratic functions of the distance from the center. Applications

[1] W. D. MacMillan, Transactions of the American Mathematical Society, vol. 11 (1910) and F. R. Moulton, *Periodic Orbits* (Carnegie Institution of Washington, Publication No. 161), chapter 4.

will be made to Jupiter and V, Mars and the two satellites, and to the Earth and Moon.

Density and Mass of an Oblate Spheroid

We assume the following formula for the density:

$$\rho = \rho_0\left[1 - \frac{\beta}{b^2}(1 - \mu^2\cos^2\phi)r^2 + \frac{\gamma}{b^4}(1 - \mu^2\cos^2\phi)^2r^4\right], \tag{2}$$

where ρ_0 denotes the density at the center, b the polar radius, μ the eccentricity of the meridian cross sections, ϕ the latitude of a point, and r its distance from the center, while β and γ are parameters whose values may be determined in each application. The polar equation of all meridian cross sections of a concentric similar shell is

$$r^2 = \frac{c^2}{(1 - \mu^2\cos^2\phi)},$$

where c is the polar radius. The density at this shell is therefore[2]

$$\rho = \rho_0(1 - \beta z^2 + \gamma z^4), \tag{3}$$

where $z = c/b$ and for our physical problem, $0 \leqq z \leqq 1$. At the surface of the planet, denoting the density by σ, we have

$$1 - \beta + \gamma = \frac{\sigma}{\rho_0}. \tag{4}$$

The mass of the planet with density (2) is obtained at once from

$$M = \int_{-\pi/2}^{\pi/2}\int_0^r\int_0^{2\pi}\rho r^2\cos\phi\, d\phi\, dr\, d\theta,$$

where θ is the longitude and the upper limit on the second integration is

$$r = b(1 - \mu^2\cos^2\phi)^{-1/2}.$$

We thus find

$$M = \frac{4\pi\rho_0 b^3}{1-\mu^2}\left(\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma\right).$$

But from the observations of the planets, we know their total mass and dimensions and consequently their average density δ, i.e.,

$$M = 4\pi\delta b^3/3(1 - \mu^2).$$

Equating the two values of M, we obtain

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[2] This expression is in harmony with Eddington's work on polytropic gas spheres contained in his book, *The Internal Constitution of the Stars* (Cambridge). From the work of Emden, he develops the density in infinite series in the distance from the center. He shows that the coefficient of the first power vanishes but it is not difficult to prove that no odd power occurs in his series.

$$\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma = \frac{\delta}{3\rho_0}. \tag{5}$$

In physical problems we would expect $\sigma \geqq 0$, $\rho_0 > 0$, $\delta > 0$, and

$$\sigma \leqq \rho_0, \quad \delta \leqq \rho_0.$$

Thus if in equation (4), $\sigma = 0$, and again $\sigma = \rho_0$, the values of β and γ are restricted to the area in a $\beta\gamma$-plane bounded by the lines

$$1 - \beta + \gamma = 0, \quad -\beta + \gamma = 0.$$

In Fig. 1, these are the lines FH and AK respectively. Similarly, if in (5) we put $\delta = \rho_0$ and again $\rho_0 \to \infty$, the values of β and γ must lie between the lines

$$-\frac{1}{5}\beta + \frac{1}{7}\gamma = 0, \quad \frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma = 0,$$

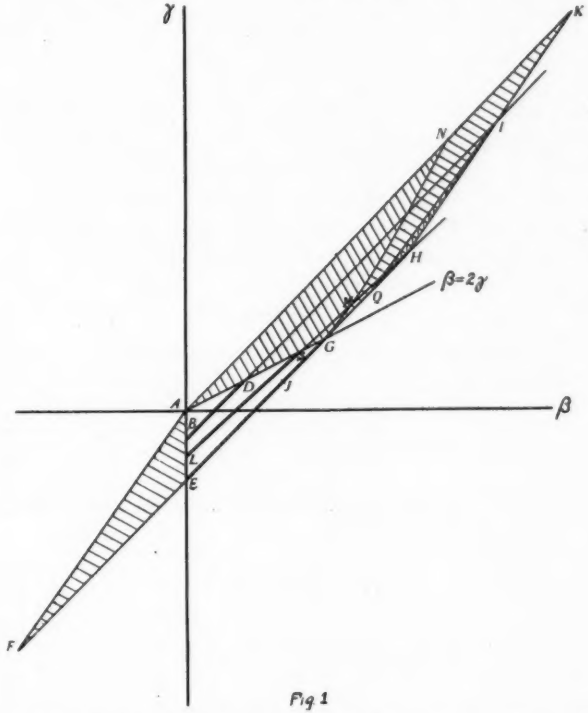

Fig 1

i.e., the lines FA and HK. The values of β and γ are therefore confined to the parallelogram $AKHF$ determined by the four lines above. The point K corresponds to an infinite density throughout the mass; F to a zero density everywhere in the body; H to an infinite density at the center and a zero density at the

surface; and A to a constant density. The points K and F must be excluded in the applications.

As we proceed along the polar axis of the planet the ratio w, of the density ρ at a point and the density ρ_0 at the center, may be written, from (3), $w=1-\beta z^2+\gamma z^4$.

The maximum and minimum values of w will occur when

$$dw/dz = -2\beta z + 4\gamma z^3 = 0, \text{ or when } z = 0, \text{ and } z^2 = \beta/2\gamma.$$

In the physical problems we would expect the greatest density to be at the center and therefore

$$d^2w/dz^2 = -2\beta + 12\gamma z^2 < 0$$

for $z=0$, or $\beta>0$.

Thus the region AEF of the parallelogram in which β is negative is excluded from consideration. The minimum value of w is

$$w = \frac{(4\gamma - \beta^2)}{4\gamma},$$

and occurs when $z^2=\beta/2\gamma$. Should this happen within the planet, i.e., when $\beta/2\gamma<1$, the corresponding values of β and γ must lie above the line $\beta=2\gamma$. But in this case, w will be positive in the area $AGMNA$, i.e., to the left of the parabola $\beta^2=4\gamma$, and negative to the right. Thus, if we exclude negative densities we must delete the area $NMGHKN$. But should we postulate that the least density of the planet occurs at the surface then $\beta/\gamma\geqq 0$, i.e., we must exclude all that portion of the parallelogram above the line $\beta=2\gamma$. The values of β and γ in the physical problem are therefore limited to the triangle $AEGA$.

Spheroid with density (2)

The potential of a homogeneous oblate spheroid on a distant unit particle is fully developed by Moulton.[1] Here we need, in addition to take into consideration the density distribution (2). The potential V then becomes

$$(6)\quad V = \frac{M}{R} - \frac{\pi}{R^3}\int_{-\pi/2}^{\pi/2}\int_0^r \rho r^4 \cos\phi d\phi dr + \frac{3\pi}{2R^5}(x^2+y^2)\int_{-\pi/2}^{\pi/2}\int_0^r \rho r^4 \cos^3\phi d\phi dr + \frac{3\pi z^2}{R^5}\int_{-\pi/2}^{\pi/2}\int_0^r \rho r^4 \sin^2\phi\cos\phi d\phi dr + \cdots.$$

In this expression we use the notation of Moulton who denotes the coordinates of the unit particle by x, y, z, with $R^2=x^2+y^2+z^2$; r is the distance of a point of the spheroid from the center and ϕ is its latitude. There is no difficulty in performing the integration with respect to r after substituting for ρ the density formula (2). Denoting the integrals in turn by A, B, C, we obtain,

[1] F. R. Moulton, *Celestial Mechanics*, p. 119.

$$A = \rho_0 b^5 F \int_{-\pi/2}^{\pi/2} (1 - \mu^2 \cos^2 \phi)^{-5/2} \cos \phi d\phi,$$

$$B = \rho_0 b^5 F \int_{-\pi/2}^{\pi/2} (1 - \mu^2 \cos^2 \phi)^{-5/2} \cos^3 \phi d\phi,$$

$$C = \rho_0 b^5 F \int_{-\pi/2}^{\pi/2} (1 - \mu^2 \cos^2 \phi)^{-5/2} \sin^2 \phi \cos \phi d\phi,$$

where $F = \frac{1}{5} - \frac{1}{7}\beta + \frac{1}{9}\gamma$. It is evident at once that $A - B = C$. We find that

$$A = \rho_0 b^5 F \frac{2(3 - \mu^2)}{3(1 - \mu^2)^2}, \qquad B = \rho_0 b^5 F \frac{4}{3(1 - \mu^2)^2}.$$

The expression for C is:

$$C = \rho_0 b^5 F \frac{2}{3(1 - \mu^2)}.$$

To write the potential (6) simply, let us use the abbreviation, $E = \frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma$. Then

$$M = \frac{4\pi\rho_0 b^3}{1 - \mu^2} E,$$

and

$$V = \frac{M}{R} - \frac{2\pi\rho_0 b^5 F}{3R^3} \cdot \frac{3 - \mu^2}{(1 - \mu^2)^2} + \frac{2\pi\rho_0 b^5 F(x^2 + y^2)}{R^5} \cdot \frac{1}{(1 - \mu^2)^2} + \frac{2\pi\rho_0 b^5 F z^2}{R^5} \cdot \frac{1}{1 - \mu^2} + \cdots.$$

Including second order terms in the expression for the potential, this becomes,

$$V = \frac{M}{R}\left[1 + \frac{b^2}{6R^4} \cdot \frac{F}{E}(x^2 + y^2 - 2z^2)\mu^2 + \cdots\right].$$

Moulton's potential function is obtained by putting, $\beta = \gamma = 0$, i.e., $F = \frac{1}{5}$, $E = \frac{1}{3}$. With this expression for the potential, by substitution in MacMillan's work we find for the motion of the line of apsides:

$$\alpha = 360^\circ\left[\frac{b^2}{2a^2} \cdot \frac{F}{E}\mu^2(1 + e^2 + \cdots)\right], \tag{7}$$

where a, b, μ and e have the same significance as before.

Pencils of Apsidal and Density Lines

Now in equation (7), if we let

$$\frac{F}{E} = p = \frac{\alpha a^2}{180b^2\mu^2(1 + e^2 + \cdots)},$$

it may be interpreted as a pencil of lines (apsidal lines)

$$\frac{1}{5} - \frac{1}{7}\beta + \frac{1}{9}\gamma = p\left(\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma\right),$$

with pole at $I(14/3, 21/5)$ and parameter p. The particular line of the pencil in each problem is determined by the value of p or, what amounts to the same thing, by the α that is assumed for a satellite. Similarly if we put $q = \delta/3\sigma$ in equations (4) and (5), they combine to form a pencil of lines (density lines)

$$\frac{1}{3} - \frac{\beta}{5} + \frac{\gamma}{7} = q(1 - \beta + \gamma), \tag{8}$$

with pole at $H(10/3, 7/3)$ and parameter q. The values of β and γ in a particular problem can now be found as the coordinates of the intersection of an apsidal line and a density line.

We distinguish three cases: viz., (a) when the distribution of density within the planet contributes to the regression of the line of apsides, i.e., $\alpha < 0$; (b) when the distribution has no effect, i.e., $\alpha = 0$ and (c) when the density within the planet causes the apses to advance, i.e., $\alpha > 0$; These three situations are distinguished according as $p < 0$, $p = 0$, $p > 0$. For $p = 0$, the apsidal line is $\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma = 0$, which is the equation of IQ (Fig. 1). Thus if β and γ are the coordinates of points on this line the corresponding density of planet would permit the satellite to revolve in a stationary ellipse. For $p < 0$, the apsidal line would pass through the area QIH. But the density of the planet within this area somewhere becomes negative. We thus doubly exclude the area.

For $p > 0$, the apsidal lines would lie in the obtuse angle KIQ. But our interest in them is confined to those lines of the pencil which cross the triangle $AGEA$, i.e., $\frac{3}{8} \geqq p \geqq \frac{1}{3}$.

Application to Jupiter and Satellite V

We shall assume that the surface density of Jupiter is negligible, i.e., $\sigma = 0$, and therefore $q \to \infty$. Thus for Jupiter, β and γ are the coordinates of points on the segment EG of the line $1 - \beta + \gamma = 0$. The value of p is therefore confined to the inequality $7/15 \geqq p \geqq 1/3$.

For the fifth satellite of Jupiter, according to H. Struve, the advance of the line of apsides is 916° per year. As the period is $11^h\ 57^m\ 22.^s7$, the advance per revolution is $\alpha = 1.^\circ2494$. With $\mu = .35888$, $a = 112{,}600$, $b = 41{,}394.37$ and $e = .0028$, we find $p = .3988$. The corresponding point on the line EG is J. Thus the

density of Jupiter may be such as to account for more or less or exactly the observed motion of the line of apsides of the satellite.

The mean density of Jupiter is $\delta = 1.34$ and from (5) the values of the central density are readily computed for the points E, J, and G. The laws of density distributions corresponding to these three points are then found to be, respectively,

$$\rho_E = 2.345(1 - z^4),$$
$$\rho_J = 4.135(1 - 1.443z^2 + .443z^4),$$
$$\rho_G = 5.863(1 - 2z^2 + z^4).$$

These densities are represented graphically in Fig. 2 by the curves marked E, J, and G. The range of possible density on any spheroidal shell of Jupiter lies between the ordinates of the curves E and G where the abscissa is the fractional part of the polar radius.

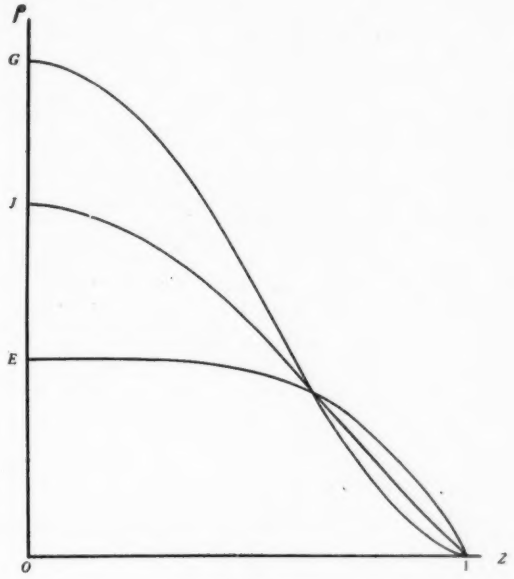

FIG. 2

To gain a clearer concept of the distribution of mass in these problems let us consider the case in which the density accounts for the whole of the motion of the apses of Satellite V. When the density of the planet of polar radius b is considered constant, its mass is

$$M_b = \frac{4\pi\delta b^3}{3(1 - \mu^2)}.$$

The ratio of this mass to a similar one of polar radius c is

$$\frac{M_c}{M_b} = \frac{c^3}{b^3} = z^3.$$

Thus an oblate spheroid of constant density and unit polar radius is divided into four equal masses by spheroids of polar radii

$$z = .6300, \quad z = .7937, \quad z = .9086 \quad \text{and} \quad z = 1.$$

The meridian cross sections of such a body are drawn with the oblateness of Jupiter in Fig. 3. The broken lines correspond to the first three computed values of z. But in the case of a density distribution according to formula (3),

$$\frac{M_c}{M_b} = z^3 \frac{\frac{1}{3} - \frac{1}{5}\beta z^2 + \frac{1}{7}\gamma z^4}{\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma}. \tag{9}$$

For the values of β and γ determined by the fifth Satellite, the planet would be divided into four equal masses by spheroids of polar radii, $z = .4623$, $z = .6137$, $z = .7523$ and $z = 1$. These are presented in Fig. 3, by the full lines.

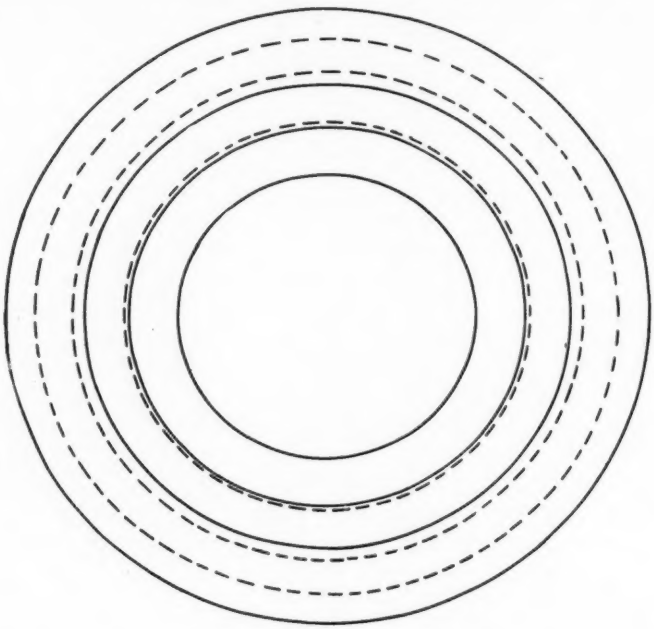

FIG. 3

Application to Mars, Deimos, and Phobos

We know the mean density of Mars, viz., $\delta = 3.96$, but its surface density is conjectural. To illustrate the theory developed, let us suppose that the motion of the line of apsides of Deimos can be accounted for by the distribution of density within the planet. The period of Deimos is $30^h\ 17^m\ 54.^s9$ and the apses complete a revolution in about 56 years; thus $\alpha = .^\circ02204$. The other constants are $a = 14{,}580$, $b = 2096.4$, $\mu^2 = .010499$ (using Lowell's value for the oblateness as 1/190) and $e = .0031$. From these data the apsidal line for Mars is $\frac{1}{5} - \frac{1}{7}\beta + \frac{1}{5}\gamma = .5640(\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma)$. The portion BD of this line (Fig. 1) contained within the triangle $AEGA$ has as extremities (0, −.3933) and (.8120, .4060), from which it would follow that

$$2.89 \leqq \sigma \leqq 3.43, \quad \text{and} \quad 4.76 \leqq \rho_0 \leqq 5.77.$$

The period of Phobos is $7^h\ 39^m\ 13.^s851$ and[1] $a = 5826.$ and $e = .0217$. With $p = .5640$ as computed for Deimos the value of $\alpha = .^\circ1381$, or the line of apsides turns through an angle $158^\circ\ 7'\ 22''.8$ per year. This result is in close agreement with the value 158° adopted by H. Struve.[2]

Application to the Earth and Moon

The mean density of the earth is $\delta = 5.52$ and the surface density,[3] $\sigma = 2.71$. The line of apsides of the moon advances[4] $11 \times 360^\circ + 109^\circ\ 2'\ 2''.52$ per century, so that the advance per revolution is $3.^\circ0438$, and therefore $p = 9167$. The values of β and γ for the earth are the coordinates of the intersection of

$$\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma = \frac{5.52}{3 \cdot 2.71}(1 - \beta + \gamma),$$

$$\frac{1}{5} - \frac{1}{7}\beta + \frac{1}{9}\gamma = 9167\left(\frac{1}{3} - \frac{1}{5}\beta + \frac{1}{7}\gamma\right),$$

the density and apsidal lines respectively. To the degree of accuracy of the computations, $\beta = 10/3$ and $\gamma = 7/3$, i.e., the point is located at H of the parallelogram of limitation. As a matter of fact more accurate computations show that the point would lie a little to the right above the point H and outside the

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[1] The values of the constants in this paper are mostly taken from the appendix to vol. I of Russell, Dugan, and Stewart, *Astronomy* (Ginn & Co.); but there is evidently a misprint in the eccentricity of the orbit of Phobos.

[2] H. Struve, Memoires de L'Academie de St. Petersburg, vol. 8. The corrections of Hall and Bower, Astronomical Journal, No. 873, do not affect the results of the present study.

[3] Hill uses $\delta = 5.67$ and $\sigma = 2.7$ in his Memoir No. 44, *On the interior constitution of the earth as respects density.*

[4] The constants for application to the Earth and Moon are taken mostly from Brown's *Tables of the Motion of the Moon.* It is interesting historically to observe that he adopts a value for the motion of the lunar perigee almost identical to that computed by Airy as referred to in Delaunay *Note sur les mouvements du périgée et du noeud de la Lune* in Comptes Rendus, vol. 74, p. 17.

parallelogram. The corresponding central density would be very large numerically and negative. Thus we find our result in harmony with the well known fact that the total motion of the lunar perigee is not due to the ellipticity and density distribution of the earth. The coordinates of the extreme points L and S on the density line within the triangle $AEGA$ for the earth are $(0, -.6447)$ and $(1.6386, .8193)$ and therefore the density ρ_0 at the center of the earth is limited to the inequality, $7.63 \leqq \rho_0 \leqq 14.99$. The two extreme values of p are $p=.6281$ and $p=4.6429$. We thus find that due to the earth the advance in the line of apsides per revolution of the moon is limited to

$$°.00021 \leqq \alpha \leqq °.00154,$$

i.e., the lunar perigee is advanced between 0.°2788 and 2.°0608 per century due to the ellipticity and density distribution of the earth.

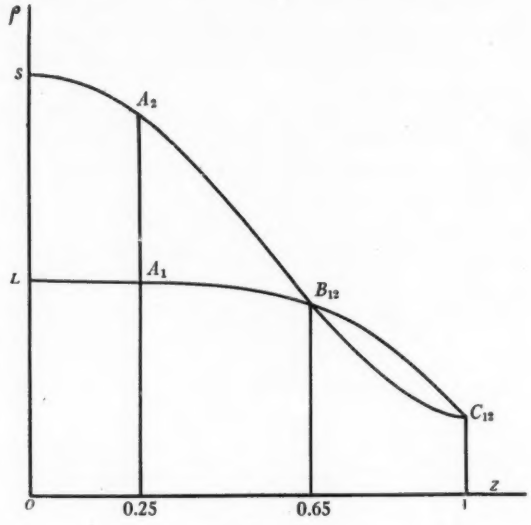

FIG. 4

We have now found that the density at a point of the polar axis of the earth or on the surface of the concentric spheroid through the point is confined between the values of ρ secured from

$$\rho_L = 7.63(1 - .6447z^4), \tag{10}$$

$$\rho_S = 14.99(1 - 1.6386z^2 + .8193z^4), \tag{10'}$$

by substituting for z, the distance measured from the center with the polar radius as unity. These equations are represented graphically in Fig. 4 by the curves $A_1B_{12}C_{12}$ and $A_2B_{12}C_{12}$, respectively. The density at a point, say $z=.25$,

is then numerically between the ordinates of A_1 and A_2. The extreme densities are equal at B_{12}, i.e., $\rho = 6.75$ at $z = .65$, and at C where $\rho = \sigma = 2.71$ at $z = 1$.

When the values of β and γ used in (10) and (10′) are substituted in (9), it is found that in the case (10), one quarter of the Earth's mass is contained within a spheroid of polar radius .58 and in the case (10′), the same part of the mass has a radius .50.

The Density at $z = \sqrt{\frac{3}{7}}$.

In Fig. 2 the density curves for Jupiter have two common points, i.e., they give a common density at the surface and at one interior point. The same is true in Fig. 4. That this is always true may be seen as follows. Consider two points (β_1, γ_1) and (β_2, γ_2) of any density line, i.e.,

$$(11)\qquad \begin{aligned} 1 - \beta_1 + \gamma_1 &= \frac{3\sigma}{\delta}\left(\frac{1}{3} - \frac{1}{5}\beta_1 + \frac{1}{7}\gamma_1\right), \\ 1 - \beta_2 + \gamma_2 &= \frac{3\sigma}{\delta}\left(\frac{1}{3} - \frac{1}{5}\beta_2 + \frac{1}{7}\gamma_2\right). \end{aligned}$$

The corresponding densities along the polar axis are

$$\rho = \frac{\sigma}{1 - \beta_1 + \gamma_1}(1 - \beta_1 z^2 + \gamma_1 z^4),$$

$$\rho = \frac{\sigma}{1 - \beta_2 + \gamma_2}(1 - \beta_2 z^2 + \gamma_2 z^4).$$

These densities will be equal when

$$\frac{1 - \beta_1 z^2 + \gamma_1 z^4}{1 - \beta_1 + \gamma_1} = \frac{1 - \beta_2 z^2 + \gamma_2 z^4}{1 - \beta_2 + \gamma_2}.$$

The roots of this equation are $z^2 = 1$, corresponding to the surface of the planet and

$$z^2 = \frac{(\beta_1 - \beta_2) - (\gamma_1 - \gamma_2)}{(\gamma_1 - \gamma_2) + (\beta_1\gamma_2 - \beta_2\gamma_1)}.$$

From equations (11) we readily find,

$$\beta_1\gamma_2 - \beta_2\gamma_1 = \frac{5(\delta - \sigma)}{3\sigma - 5\delta}(\gamma_1 - \gamma_2),$$

and

$$\beta_1 - \beta_2 = \frac{5}{7}\cdot\frac{7\delta - 3\sigma}{5\delta - 3\sigma}(\gamma_1 - \gamma_2),$$

from which the second root is reduced to $z^2 = \frac{3}{7}$. Thus with the distribution (2), the density of planets at $z = \sqrt{\frac{3}{7}}$, is the same for all points of a density line and may be determined with the accuracy of the surface density.

In this study, the computations have been made and the figures drawn by Miss Gretchen Mullison, Research Assistant.

SOME PROPERTIES OF CORRELATIVE VERTEX LINES IN A PLANE TRIANGLE

By VLADIMIR KARAPETOFF, Cornell University

Theorem 1. *The three correlative vertex lines in a triangle always intersect in one point.*

Proof: By definition, let a straight line passing through the vertex of a triangle be called a vertex line. Let α, β and γ be three arbitrary scalar parameters associated with the sides opposite the vertices A, B, and C, respectively. Let the direction AL be determined in the following manner: Take $AK = AB/\gamma$ and $AM = AC/\beta$; construct a parallelogram on these lines and let its diagonal be called AL. If now a line from B and a line from C be constructed in the same manner, using the three quantities α, β, and γ in the cyclic order, then these three lines by definition are called correlative. It is required to prove that any three correlative lines intersect in one point, O. Several well-known theorems may be deduced as special cases of this general theorem.

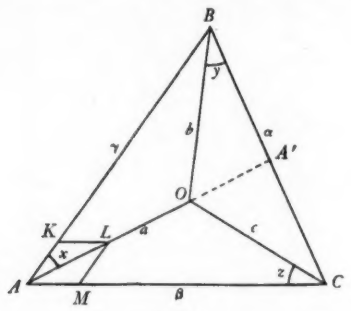

FIG. 1

Instead of starting with a given triangle we begin with an origin O, and with three arbitrary vectors, $OA = \mathbf{a}$, $OB = \mathbf{b}$, and $OC = \mathbf{c}$, drawn from O. These vectors determine the triangle, and the condition that the three vectors lie in the same plane is

$$\mathbf{a}\alpha + \mathbf{b}\beta + \mathbf{c}\gamma = 0, \tag{1}$$

where α, β, and γ are known quantities for given $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Of course, α, β, and γ may all be multiplied by some arbitrary scalar quantity without changing the problem, because it is their ratio only that determines the triangle.

It will be proved now that α, β, γ, defined by (1), are identical with the coefficients defined in Fig. 1. For this purpose add $(-\beta\mathbf{a} - \gamma\mathbf{a})$ to both sides of equation (1) and transfer $\mathbf{a}\alpha$ to the right. Then equation (1) becomes:

$$(\mathbf{b} - \mathbf{a})\beta + (\mathbf{c} - \mathbf{a})\gamma = -\mathbf{a}(\alpha + \beta + \gamma)$$

or

$$(\mathbf{b} - \mathbf{a})/\gamma + (\mathbf{c} - \mathbf{a})/\beta = -\mathbf{a}(q/\beta\gamma), \tag{2}$$

where $q = \alpha + \beta + \gamma$.

But $(\mathbf{b} - \mathbf{a})/\gamma = AK$ and $(\mathbf{c} - \mathbf{a})/\beta = AM$. Thus, (2) states the fact that the vertex line AL is in the direction of the given vector $\mathbf{a}$. By cyclic rotation we can prove that the two other corresponding vertex lines coincide with the given vectors $\mathbf{b}$ and $\mathbf{c}$, and this proves the theorem.

In specific cases it is only necessary to prove that it is possible to assign definite values to α, β, and γ. Then such vertex lines will intersect in one point.

Special Cases

(a) *Medians.* Here $\alpha = \beta = \gamma$, so that the medians intersect in one point. Incidentally, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ being equal to zero, we thus prove that it is always possible to construct a new triangle using the medians of a given triangle as its sides.

(b) *Bisectors of the angles.* Here α, β, γ, are proportional to the lengths of the corresponding sides of the triangle. Hence the bisectors intersect in one point. Equation (1) in this case expresses an intrinsic relationship involving only the lengths of the sides and the distances from the vertices to the center of the inscribed circle.

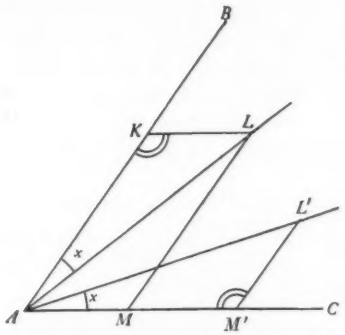

FIG. 2

(c) *Coffin's case.* Let O be an arbitrary point within the triangle and let the vertex lines form angles x, y, z with the sides, as shown in Fig. 1. Let now three other vertex lines be drawn in such a way as to form the same angles with the adjacent sides; for example, the new vertex line from A forms an angle x with AC instead of AB (Fig. 2). It is required to prove that the three new vertex line intersect in one point. This theorem was suggested by Dr. J. G. Coffin as an exercise in vector analysis, in a private letter to the writer.

Since by supposition the first three vertex lines intersect in one point, they form a corresponding set, so that for them there are definite values of α, β,

and γ. It is only necessary to prove that there are definite values, α', β', γ', for the second set. Let AL (Fig. 2) be the given vertex line for which by assumption, $AK=AB/\gamma$ and $AM=AC/\beta$. Let AL' be the new vertex line so that $M'L'=AB/\gamma'$ and $AM'=AC/\beta'$. From the similar triangles we get

$$\frac{AC/\beta'}{AB/\gamma'}=\frac{AB/\gamma}{AC/\beta},$$

or

$$AB^2/(\gamma\gamma')=AC^2/(\beta\beta').$$

Thus, the ratio of β' to γ' is determined by the ratios β/γ and AB/AC. Similarly, for the vertex B we can write $AB^2/(\gamma\gamma')=BC^2/(\alpha\alpha')$, so that the ratio of α' to γ' is also known. Therefore, the three new lines also form a corresponding set and consequently intersect in one point.

(d) *The altitudes.* In this case $\boldsymbol{a}$ is perpendicular to $(\boldsymbol{b}-\boldsymbol{c})$, so that $(\boldsymbol{b}-\boldsymbol{c})\cdot\boldsymbol{a}=0$, or $\boldsymbol{b}\cdot\boldsymbol{a}=\boldsymbol{c}\cdot\boldsymbol{a}$, and by analogy $\boldsymbol{b}\cdot\boldsymbol{a}=\boldsymbol{b}\cdot\boldsymbol{c}$.

Let this product of the two segments of each altitude be equal to k^2, so that $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{c}=\boldsymbol{c}\cdot\boldsymbol{a}=-k^2$.

Form a scalar product of equation (1) with $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. The following three equations are then obtained:

$$\alpha a^2-(\beta+\gamma)k^2=0,\quad \beta b^2-(\gamma+\alpha)k^2=0,\quad \gamma c^2-(\alpha+\beta)k^2=0.$$

The quantity k^2 may be eliminated from these equations and the resultant two equations solved for α/γ and β/γ. Thus, there are definite ratios for α, β, γ, for the altitudes of a given triangle, so that the directions of the altitudes form a set of corresponding vertex lines and therefore intersect in one point. This proposition also follows directly from the next theorem.

Theorem 2. *Any three correlative vertex lines divide the opposite sides of the triangle into parts proportional to the parameters α, β, γ, belonging to the adjacent sides.*

Proof: Referring to Fig. 1, let A' be the intersection of the vertex line AO with the side BC; and let $m=BA'/A'C$. By a cyclic substitution, for the other two vertex lines and sides let $n=CB'/B'A$ and $p=AC'/C'B$.

Then, for the triangle OBA' we have:

$$OA'=-s\boldsymbol{a}=\boldsymbol{b}+(\boldsymbol{c}-\boldsymbol{b})m/(1+m)$$

where s is a scalar. Hence,

$$s\boldsymbol{a}+\boldsymbol{b}/(1+m)+\boldsymbol{c}m/(1+m)=0. \tag{3}$$

Comparing this expression with equation (1) we find that $m=\gamma/\beta$. Analogously, $n=\alpha/\gamma$ and $p=\beta/\alpha$, which proves the proposition.

Comparing the coefficients of $\boldsymbol{a}$ and $\boldsymbol{b}$ in equations (3) and (1) we find that $\alpha/s=\beta(1+m)$, or using expression $m=\gamma/\beta$ for m, $s=\alpha/(\beta+\gamma)$.

For medians, $\alpha=\beta=\gamma$; $m=n=p=1$; and $s=1/2$; hence the medians trisect each other. For bisectors of angles, α, β, γ are proportional to the lengths of the corresponding sides, so that the equation $m=\gamma/\beta$ expresses the familiar

theorem that the bisector of an angle divides the opposite side into parts proportional to the other two sides.

From $m=\gamma/\beta$, $n=\alpha/\gamma$, $p=\beta/\alpha$, we have $mnp=1$, which means that any three vertex lines intersect in one point if they divide the opposite sides in ratios the product of which is equal to unity. Thus, for the altitude from vertex A, $m=AB\cos B/AC\cos C$.

Writing similar expressions for the other two sides, we find that m, n, and p satisfy the equation $mnp=1$, so that the three altitudes intersect in one point.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. A TRIPLY COMPOSITE HOMOGENEOUS POLYNOMIAL

By E. T. BELL, California Institute of Technology

1. From a paper in the Transactions of the American Mathematical Society for April, 1929, I transcribe the following definition of composition, as stated by Professor O. C. Hazlett (who gives full references to Dickson and other writers.)

"If $f(x)$ is any homogeneous polynomial in the n variables $x_1 \cdots, x_n$, such that

$$f(x)f(\xi) = F(X), \tag{1}$$

where the X's are bilinear functions of the x's and ξ's,

$$X_k = \sum_{i,j}\gamma_{ijk}x_i\xi_j \quad (k = 1, \cdots, n), \tag{2}$$

then $f(x)$ is said to admit the composition (2)."

Readers of Professor Hazlett's illuminating paper will note that in the proofs of the theorems, as in the fundamental papers by Dickson, two conditions are essential: $f(x)$ shall not be expressible in fewer than n variables; the functions X effecting the composition shall be bilinear in the variables of the forms compounded.

Some years ago I worked out a considerable part of a theory of composition in which both of these conditions are abandoned. The complete theory in this extended sense can be given when and only when the problem implied in (1), (2) is completely solved, which has not yet been done. In the meantime it may be of interest to give the results in the algebraic number case, which are fairly complete; this I hope to do in a paper to appear elsewhere. All of the specific

instances, except in the simplest case, involve very long formulas when written out fully; hence no examples of the straightforward general theory in the algebraic case are included in the paper mentioned. It may therefore be of interest to state the formulas for the simplest case of what I have called multiple composition, which is as follows.

If n is a composite integer >2, which has a divisor $d \geqq 2$, we consider those homogeneous polynomials of degree n in n independent variables having the following properties:

(A) The polynomials admit composition in the sense (1), (2), and are not expressible by a linear transformation on the variables, in fewer than n variables.

(B) Each of the polynomials can be written as a homogeneous polynomial in d variables; the d new variables are homogeneous polynomials, with rational coefficients, in the original variables, and are of degree n/d in those variables.

(C) The same as (B), with d, n/d interchanged throughout.

(D) Each of the polynomials in (B), (C), considered as a function of the new variables, has the property (A).

(E) The generalization of $(A)-(D)$ in which $n=n_1n_2\cdots n_s$, $n_1>1$, $n_2>1, \cdots, n_s>1$. There are then 2^s-1 compositions of the kind described.

It will be sufficient to state the formulas for the case $s=2$, $n_1=n_2=2$. Let the u's, v's, x's, y's, z's, w's be independent variables, and p, q, r, s parameters. Write

$$g(u,v) \equiv u^2 + puv + qv^2, \quad h(u,v) \equiv u^2 + ruv + sv^2;$$

$$Q(u_1,v_1,u_2,v_2) \equiv u_1u_2 - qv_1v_2,$$

$$P(u_1,v_1,u_2,v_2) \equiv u_1v_2 + v_1u_2 + pv_1v_2,$$

$$S(u_1,v_1,u_2,v_2) \equiv u_1u_2 - sv_1v_2,$$

$$R(u_1,v_1,u_2,v_2) \equiv u_1v_2 + v_1u_2 + rv_1v_2.$$

Let a suffix x, y, z, or w denote partial differentiation with respect to the suffix, thus $g_z(z,w) \equiv \partial g(z,w)/\partial z$, etc. Write

$$G'(x,y,z,w) \equiv g(x,y) - sg(z,w),$$

$$G''(x,y,z,w) \equiv xg_z(z,w) + yg_w(z,w) + rg(z,w),$$

$$H'(x,y,z,w) \equiv h(x,z) - qh(y,w),$$

$$H''(x,y,z,w) \equiv xh_y(y,w) + zh_w(y,w) + ph(y,w).$$

Let a suffix 1, 2, or 3 on any of the above functions refer to variables x_i, y_i, z_i, $w_i (i=1,2,3)$; thus $G'_i \equiv G'(x_i,y_i,z_i,w_i)$, etc. Write

$$x_3 \equiv x_1x_2 - qy_1y_2 - sz_1z_2 + qsw_1w_2,$$

$$y_3 \equiv x_1y_2 + y_1x_2 + p(y_1y_2 - sw_1w_2) - s(z_1w_2 + w_1z_2),$$

$$z_3 \equiv x_1z_2 + z_1x_2 + r(z_1z_2 - qw_1w_2) - q(y_1w_2 + w_1y_2),$$

$$w_3 \equiv x_1w_2 + w_1x_2 + y_1z_2 + z_1y_2 + p(y_1w_2 + w_1y_2) + r(z_1w_2 + w_1z_2) + prw_1w_2.$$

Then, as may be verified directly,

$$\text{(I)} \qquad G'_3 = S(G'_1, G''_1, G'_2, G''_2), \qquad G''_3 = R(G'_1, G''_1, G_2', G_2''),$$

$$H_3' = Q(H_1', H_1'', H_2', H_2''), \quad H_3'' = P(H_1', H_1'', H_2', H_2'');$$

$$\text{(II)} \quad g(H'(x,y,z,w), H''(x,y,z,w)) = h(G'(x,y,z,w), G''(x,y,z,w)).$$

Denote either of the functions in (II) by $f(x, y, z, w)$. Then $f(x, y, z, w)$, homogeneous of degree 4 in x, y, z, w, has the triple composition as indicated below.

$$\text{(III)} \qquad \begin{aligned} f(x_1,y_1,z_1,w_1)f(x_2,y_2,z_2,w_2) &= f(x_3,y_3,z_3,w_3), \\ &= g(H_1',H_1'')g(H_2',H_2'') = g(H_3',H_3''), \\ &= h(G_1',G_1'')h(G_2',G_2'') = h(G_3',G_3''). \end{aligned}$$

Examples in which the subsidiary forms $g, h, \cdots$, are of different degrees are perhaps more striking. The simplest is that in which g is of degree 2, h of degree 3. As the formulas of composition are rather long, we shall not transcribe them. The above is, so far as I know, the first instance in the literature of a multiply composable form. The situation defined in $(A)-(E)$ is here stated for the first time.

II. A Theorem About Maximum and Minimum Points

By Carl Gundersen, Oklahoma A. and M. College

Theorem: *If $f(x)$ is a continuous function with continuous derivatives, and if $f'(x')=0$, and $f''(x')=0$ at the same time as it, $[f''(x')]$, is a minimum, then $f(x')$ is also a minimum.*

Proof: Since $f''(x)$ has a minimum point at x', and $f''(x')=0$, $f''(x)$ must be positive near $x=x'$. That means that $f'(x)$ is an increasing function both before and after the point $x=x'$, and since $f'(x')=0$, $f'(x)$ is negative *before*, and positive *after* the point $x=x'$; it follows that $f(x)$ is decreasing *before* and increasing *after* the point $x=x'$, which is only another way of saying that $f(x')$ is a minimum point.

This theorem is useful in discussing the case when $f''(x')=0$. Assume that $f^{(n)}(x')$ is positive, and $f^{(n-1)}(x')$, $f^{(n-2)}(x')$, $\cdots$, $f'(x')$ are all zero; then $f^{(n-2)}(x')$ must be a minimum by a well known theorem; $f^{(n-4)}(x')$ must be a minimum by the theorem proved above; again $f^{(n-6)}(x')$, $f^{(n-8)}(x')$, etc. must be minima. Therefore if n is even, $f(x')$ is a minimum; if n is odd, $f'(x')$ is a minimum, and being equal to zero it is positive near $x=x'$, and $f(x)$ must have a point of inflection at $x=x'$.

An analogous theorem holds of course for a maximum.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Modern Geometry. By Roger A. Johnson. Houghton Mifflin Company, 1929. xiii+319 pages. $3.50.

This is a very interesting book both as regards its content and the manner of presentation. The following chapter headings give a fair idea of the material covered. I, Introduction; II, Similar figures; III, Coaxal circles and inversion; IV, Triangles and polygons; V, Geometry of circles; VI, Tangent circles; VII, The theorem of Miquel; VIII, Theorems of Ceva and Menelaus; IX, Three notable points (orthocenter, circumcenter, and median point of a triangle); X, Inscribed and escribed circles; XI, The nine point circle; XII, Symmedian point and other notable points (isogonic centers, Nagel point); XIII, Triangles in perspective; XIV, Pedal triangles and circles; XV, Shorter topics (cyclic quadrangle, Morley's theorem, etc.); XVI, The Brocard configuration; XVII, Equibrocardal triangles; XVIII, Three similar figures.

We are entirely in sympathy with the aim of this book, which is mainly to give beginners in mathematics something interesting to think about. It is rather the fashion to pooh-pooh this branch of mathematics and to say that the time spent on it could be better devoted to matters of a more arithmetical nature. But it is a commonplace that mathematics is like art, and to really appreciate art one must first get interested in it; the main object of a secondary-school or junior-college teacher who believes in his work is to encourage an interest and curiosity in his students, the actual material he talks about being relatively unimportant. We feel that the book in hand is admirably calculated to arouse and stimulate such an interest. Furthermore the careful formulation of proofs so that they are valid for all types of figures involved furnishes a good example to the beginner of what mathematical reasoning should be.

A reviewer is, we understand, expected to point out certain features of the book in question which are, in his opinion, undesirable. In this case this is a little difficult. There might possibly be more exercises, and we prefer the name Fermat points to isogonic centers and the name Hessian points to isodynamic points. There is an easily supplied omission in the statement of the theorem on p. 221. On the whole we heartily recommend the book and wish it the success it deserves.

FRANCIS D. MURNAGHAN

Gerhard Hessenberg's Vorlesungen über darstellende Geometrie, herausgegeben von E. Salkowski, o. Professor an der technischen Hochschule Charlottenburg. Leipzig, Akademische Verlagsgesellschaft, 1929. 274 pages, 481 figures.

Descriptive Geometry. By Harvey Herbert Jordan and Francis Marion Porter, University of Illinois. Ginn and Co., Boston, 1929. xi+349 pages, 791 figures.

The first book introduces its subject with a practical explanation of the use of drawing instruments, followed by three parts of about equal length, on axonometry, the two picture (h, v) method, and curves and surfaces. The sequence is unusual, but the essential features of each method are clearly and concisely presented, and applied. For an American student it would have been improved by a larger supply of unsolved exercises. The press-work and drawings are excellently well done, and the tone of a thoroughly scientific work is preserved throughout.

The second book presents the subject in accordance with the better traditional American methods, presenting a large number of principles, without putting great emphasis on any, pointing out that different ones should be used for different purposes. The first third of the volume is devoted to the ordinary rectilinear problems in h, v, p presentation; then the principles are applied to curved surfaces. A short well written chapter treats of pictorial drawing, including isometric and perspective representations. The book closes with an excellent list of over 400 exercises, arranged for use as each chapter is completed. As a guide to the subject as an art, this book is particularly well adapted; as an introduction to the science of descriptive geometry, the former is preferable.

VIRGIL SNYDER

Vorlesungen über Differential- und Integralrechnung, Bd. II. Funktionen mehrerer Veränderlichen. By R. Courant. Julius Springer, Berlin, 1929. vii+360 pages.

The second and concluding volume of Courant's lectures on the calculus, devoted to functions of several variables, follows the same plan as the first volume[1] and has the same outstanding merits. To borrow from a review by von Mises, "In Courant we have an author · · · who risks the statement of theorems which do not give the impression to the reader that each application of them is attended by the gravest dangers. In contrast with the present-day style of mathematical writing, in which the essential content is everywhere hidden behind a chaos of secondary details, the book of Courant breathes an entirely different spirit." One should not judge from this that the author ever departs from strict rigor; rather he chooses to state a theorem in a usable form rather than in its utmost generality and refinement. Thus the theorem that $f_{yx}=f_{xy}$ is proved under the hypothesis that both these derivatives are con-

[1] Reviewed in this Monthly, vol. 36 (1929), pp. 96–98.

tinuous in a region; a footnote then shows that we need only assume that one mixed derivative exists and is continuous at the point in question. Again matters of considerable detail or refinement are usually placed in the *Anhänge* that follow each chapter.

The book opens with an introductory chapter on three-dimensional analytical geometry and vector analysis; this is to be studied by the reader as the matters treated therein are needed. The following chapters deal with partial and total derivatives (II); further developments and applications of the same (III); integrals over regions (IV); curve integrals, and the integral theorems of Gauss, Stokes, Green and their significance (V); applications, mostly involving differential equations (VI). The book concludes with a 27-page summary of the principal formulas and theorems in both volumes.

For many theorems both scalar and vector formulations are given. To quote Courant, "Many facts and relations of the differential and integral calculus take on an essentially clearer and simpler aspect when the concepts and notation of vector analysis are employed." The Gibbs cross is used for the vector product, juxtaposition for the scalar. This usage is apparently gaining ground in Germany, being used in recent books by Runge, Blaschke, von Mises, and Fraenkel.

While the author never departs from precise mathematical formulation, interpretation, comparison, and application are constantly in the foreground. These lectures, therefore, should be of great service to all who wish to have a thoroughly modern grasp of the Calculus or who wish to apply it fruitfully in other fields.

LOUIS BRAND

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung. Von Herman Grassmann. Zweiter Band: Ternäres. Zweiter Teil. Mit 259 Figuren im Text. Leipzig und Berlin, B. G. Teubner, 1927. xvi+522 pp.

Increased attention has lately been given the Ausdehnungslehre. This has been due in part to favorable expressions of opinion concerning this subject by mathematicians of renown and in part to the close relation which exists between Grassmann's geometrical algebra and the so-called absolute calculus of Ricci, made prominent by the theory of relativity. It is therefore very fortunate that this interest should be further stimulated by a treatise on projective geometry, based on the Ausdehnungslehre, by Grassmann's son. The present volume is in continuation of two previous volumes by the same author which appeared in 1909 and 1913 respectively.

The principal algorithms on which the calculus of Grassmann is based are the "combinatorial" and the "algebraic" multiplications. The second part of the Ausdehnungslehre of 1862 discusses products of the latter type. "Algebraic" multiplication is relevant to the theory of extensive functions. The simplest case of an extensive function, namely, an integral function of the first degree, leads to the conception of an extensive quotient or lacunary expression with one

space which includes as particular case the matrix as symbolic operator. Geometrically, the importance of an extensive quotient lies in the fact that the concepts of collineation and reciprocity are readily expressed in terms of it. This was explicitly pointed out by Hermann Grassmann[1] the elder in the Ausdehnungslehre of 1862. As one of the editors of Grassmann's works, the son Hermann added an elaborate note amplifying his father's remarks and Hermann Grassmann, Jr.'s studies and subsequent publications, including the volume before us, may be regarded as further expansions of this part of the Ausdehnungslehre (pp. 67–208).

The present work falls into three parts: first, projective geometry; second, "parallel" geometry; third, "right angle" geometry. Projective geometry, and parallel geometry which is treated as incidental to the former (cf. page 380), absorb 313 pages of the book; pages 379–522 are devoted to metrical geometry referred to the "circular points at infinity." In contrast with projective geometry, Grassmann conceives of metrical geometry as consisting of "parallel" geometry and "right-angle" geometry. Perpendicularity is made the basis of linear measurement. The length of a linear segment is defined in terms of a quadratic form (p. 393). Grassmann, Jr.'s treatment of the projective geometry of the plane presents a rich collection of theorems in the proofs of which researches of other investigators, using the ordinary methods of analysis, have been freely used. In particular, the author expresses in the preface his indebtedness to Gundelfinger. Such a relation of Grassmann, Jr.'s research to contemporaneous literature finds a remarkable precedent in his father's work. In fact, Grassmann, Sr. in his Ausdehnungslehre of 1844 states on page 138 (collected works): "Nun finden wir zu dem Wege den wir hier verlangen in der neueren Geometrie mannigfache Vorarbeiten, in unserer Wissenschaft aber ist uns der Weg selbst auf's Vollkommenste vorgezeichnet." Regardless, therefore, of Grassmann, Jr.'s frequent reference to other authors, his work stands for itself.

A central point in the theory of the present work is the concept of a lacunary expression which the author develops on page 171 in connection with "apolar polar systems." This development is easily understood and constitutes a valuable introduction to the theory of lacunary expressions in the Ausdehnungslehre of 1862. It is entirely logical that the algebraic product is introduced (p. 175). To the lacunary expression and the algebraic product correspond the two fundamental forms, the *lacunary form* and the *algebraic* (product) *form*; these are called by the author "Lückenform" and "Potenzform." Corresponding to the concepts "order" and "class" two fundamental theorems concerning algebraic (product) forms are established (pp. 177, 185). A further application is made to curves of the third degree (pp. 289, 378). In this connection, the author expresses his indebtedness to Durège.

In the preceding theory, the concepts *transformation*, *group* and *invariant* lie hidden but they are not explicitly developed by the author. It is of course well known that these concepts hovered before the mind of the elder Grassmann

[1] Collected Works, vol. 1–2, pp. 256–257.

and vague references to them may be found in the Ausdehnungslehre of 1844. Indeed, Grassmann's geometrical algebra forms an admirable basis for the genetic motivation of the preceding concepts. A valuable contribution in this direction has recently been made by Dr. E. Carus in his Chicago dissertation.[1] Grassmann, Jr. also refers to the "mehrfaltige Produkte" of E. Müller and the "Faltungsprozess" of Gordan in a note on page 188 of the present work. A further application of certain concepts of the Ausdehnungslehre, with special reference to the theory of invariants, has recently been made by E. Study in his vectorial "Theorie der Invarianten linearer Transformationen." However, Study apparently interprets mathematics narrowly as a mere science of number; consequently he rejects the axiomatic foundation of geometry and would subordinate the Ausdehnungslehre to the theory of invariants. Such a position is clearly opposed by the elder Grassmann and no doubt also by the author of the present work.

Hermann Grassmann, Jr. did not live to see his splendid volume in print. The publication of this work was made possible through the financial support of a group of friends, notably Dr. E. Carus of La Salle, Ill. Hermann Grassmann, Jr.'s work represents mature scholarship and is indispensable to all students interested in the important field of vectorial geometry; it should also promote research in the Ausdehnungslehre itself, whose application to geometry has been judged "the most comprehensive and above all the most natural system of geometric analysis yet discovered."

ARTHUR R. SCHWEITZER

Neue Astronomie. By Johannes Kepler. Translated and introduced by Prof. Dr. Max Caspar. R. Oldenburg, Munich-Berlin, 1929. 66+416 pages, xiii+68 figures.

This translation of Kepler represents the very commendable purpose of the translator and publisher to make the German readers better acquainted with the forceful style and colorful personality of their great countryman. There is no attempt to abridge or elaborate any of the passages. So far as possible it is Kepler's writing unembellished and unadorned.

It is also hoped that there will result a greater appreciation of Kepler's accomplishment, comparable in many ways, as he thought, with that of Columbus, and a fulfillment of his expressed wish that many would enjoy the narrative of his celestial discoveries as they had those of the geographical ones of Columbus, Magellan, etc.

The book performs a long neglected duty of doing for Kepler what had long been done for Ptolemy, Copernicus, Newton, and Descartes, for strangely enough it seems to be the only complete German edition ever published.

Preceding the translation there is a preface, an introduction, a brief outline

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[1] *Invariants as products and a vector interpretation of the symbolic method*, The Open Court Publishing Co., Chicago and London, 1927.

of planetary theory before the time of Kepler, the origin and development of the *Neue Astronomie*; and at the end of the book there are comments by the publisher and a table of contents that groups the material into five parts according to the outstanding ideas.

The book is printed on firm unglazed paper with very legible typography and is altogether a very attractive volume. The price ranges from M38.50 for the linen to M100 for the pigskin binding.

F. E. CARR

Bibliography, Practical, Enumerative, Historical; An Introductory Manual. By H. B. Van Hoesen with the collaboration of F. K. Walter. Charles Scribner's Sons, New York, 1928. Price $7.50.

American students are frequently accused of lack of erudition, perhaps not without cause. Erudition is acquired, if at all, by extensive and selective reading. At most institutions of higher learning in this country the supply of useful books is large and the conditions for consulting them most favorable. If the accusation is well founded one may consequently be justified in concluding that the students do not know when, how, or what to read. At Princeton University the assistant librarian, Dr. H. B. Van Hoesen, has found it necessary to give courses for graduate students in the use of the library. The highly interesting and entertaining book under review is an outgrowth of these courses. The book starts out with practical, much needed advice to authors. Subject bibliography takes about 85 pages which includes two pages devoted to mathematics. About 120 pages are given to the use of the library in general: library science, general reference books, special, national, and universal bibliographies. The historical bibliography takes about 150 pages devoted to the history of writing, printing, books and libraries. A bibliographical appendix and a topical index, each with about 2000 entries, occupy the rest of the book, an impressive object lesson.

EINAR HILLE

MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, University of Texas, Austin, Texas.

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Euclidean Circle of the Illinois State Normal University, Normal, Illinois.

The Euclidean Circle was organized on Dec. 14, 1927, under the leadership of Mr. C. N. Mills, professor of mathematics, and Miss Edith Irene Atkin, assistant professor of mathematics.

The officers for the year 1928–29 were as follows: Elmer J. Graber ('29)—"Major Arc"—Presiding Officer; Pauline Whipple ('29)—"Minor Arc"—Vice-Presiding Officer; Willis T. Maas ('29)—"Inscribed Polygon"—Secretary; Dorothea Concklin ('31)—"Center"—Treasurer. Miss

Edith Irene Atkin—"Circumscribed Polygon"—Acting Head of the Mathematics Department, 1928–29, was Sponsor.

The program for the college year 1928–29 was as follows:

Sep. 27, 1928. Business. Homecoming Social Committee.

Oct. 11. "The theorem of Nicomachus with extensions," by Elbert C. Parker, '30; "Magic squares," by Blanche Hinthorne, '29.

Oct. 20. First Annual Homecoming Breakfast. 8:30 A.M. Toasts: "Limits," by Christian E. Harpster, '28; "Going off on a tangent," by Blanche Hinthorne, '29; "Signs of grouping," by Clara Whitfield, '28; "Talk," by President David Felmley.

Oct. 25. Proofs of the Pythagorean theorem: "*The Pythagorean Proposition*," by E. S. Loomis and Miss Atkin; "Geometric Proofs," by Ella Iliff, '31; "Geometric and algebraic proof," by Sue Szabo, '30; "Geometric and algebraic proof," by Ella Rosenthall, '31.

Nov. 22. "Flatland," by Verna Mae Thomassen, '31; "Fourth dimension," by Arthur R. Grismer, '29.

Dec. 13. Initiation of new members. Social Hour.

Jan. 31, 1929. "Old measuring instruments," by Dorothea Concklin, '31.

Feb. 28. "Mathematics in chemistry," by Pauline Whipple, '29.

Apr. 4. "Methods of drawing a straight line by linkages," by Clyde Kaiser, '30.

May 2. "A number system with a base twelve," by Agnes Hanson, '31; "How to trisect an angle with a straight edge containing two marked points," by Elmer J. Graber, '31.

May 30. A Play: "Mock trial of B versus A, or solving a personal equation by judicial process." Adapted by Kathryn McSorley, Hunter College. Social Hours.

The officers for the year 1929–30 are: Major Arc, Clyde Kaiser ('30); Minor Arc, Verna Mae Thomassen ('31); Inscribed Polygon, Dorothea Concklin ('31); Center, W. A. Fiske ('31).

(Report by Willis T. Maas)

The Junior Mathematics Club of the University of Chicago, Chicago, Ill.

The officers for the year 1927–1928 were: B. W. Jones, President; May M. Beenken, Chairman of Committee on Program, T. F. Cope, Secretary and Treasurer.

The program for the year 1927–1928 was as follows:

Oct. 21, 1927. "The history of the Junior Mathematics Club," by Professor H. E. Slaught; "Mathematical study in Italy," by Professor E. P. Lane.

Nov. 2. "Selected topics in the calculus of variations," by Mr. T. F. Cope.

Nov. 16. "Construction of higher plane curves," by Mr. R. S. Shaw.

Dec. 7. "The Borel theorem and its applications," by Mr. E. J. McShane.

Jan. 11, 1928. "The evolution and dissolution of matter," by Professor MacMillan.

Feb. 9. "Various map projections," by Mr. C. A. Sherer.

Feb. 21. "Hyperbolic functions," by Mr. G. D. Gore.

Mar. 14. "The Lorentz transformations," by Mr. R. H. Bardell.

Apr. 11. "Sturm's theorems of comparison and oscillation for solutions of differential equations," by Mr. F. R. Bamforth.

Apr. 25. "How can graduate mathematical instruction be improved?" by everyone, especially B. J. Jones, R. S. Shaw, A. Woods.

May 9. "Geometric representation of real and complex points of elementary curves," by Mr. G. W. Spenceley.

May 23. "Various types of coordinate systems in geometry," by Dr. Jesse Douglas.

(Report by T. F. Cope)

Zeta Mu Tau, University of Washington, Seattle, Wash.

The program of Zeta Mu Tau, the mathematics club of the University of Washington, for the year 1927–1928, was as follows:

Oct. 5, 1927. "The Path of light in the gravitational field," by John Hicks, '29.
Oct. 19. "The history of our number system," by Lucile Morry, '28.
Nov. 2. "Four times," by Larned Meacham, '29.
May 17, 1928. Initiation of pledges, followed by a banquet with addresses by Dr. R. E. Moritz and Dr. R. M. Winger.

Zeta Mu Tau is essentially an undergraduate society, with graduate students and members of the Mathematics Department taking little part in its activities. It was formed for the purpose of stimulating campus interest in a very little-appreciated branch of learning, and to provide a means for the exchange of ideas among the three groups interested in mathematics, viz., the mathematics majors, the engineers, and the students of the physical sciences.

The officers for the year 1927–28 were as follows: Louis Berger, President; Martha Hardy, Vice-President; Lucile Morry, Secretary; Lucile Anderson, Treasurer.

(Report by Martha Hardy)

The Mathematics Club of the University of Colorado, Boulder, Colorado.

The program for the year 1928–29 was as follows:
Oct. 11, 1928. "Man-made science," by S. Hacker.
Oct. 25. "Impressions of the 1928 International Congress of Mathematicians, Bologna, Italy," by Associate Professor C. Kendall.
Nov. 18. "Graphical representation of complex functions," by E. Rainville.
Nov. 25. "The place of mathematics in education," by R. B. Pinson.
Jan. 17, 1929. "Circles connected with the triangle," by H. Harms.
Jan. 31. "Magic squares," by P. Folk.
Feb. 21. "Interesting theorems in modern geometry," by H. A. Miley.
Feb. 28. "Mathematical geography," by Professor C. A. Hutchinson.
Mar. 28. "Calculation of the date of Easter," by R. Remke.
Apr. 11. "Mathematics, from the points of view of the mathematician and the physicist," by L. Strait.
Apr. 25. "Majoring in mathematics," by Professor A. J. Kempner.
May 16. "Conics in a spring suit: or, conic sections in terms of the length of arc and radius of curvature," by Professor G. H. Light.
May 24. A Fry in Gregory Canyon.

The officers for 1928–29 were: Sidney Hacker, President; Earl Rainville, Vice-President; Pauline Folk, Secretary. The officers for 1929–30 are: Earl Rainville, President; Louis Strait, Vice-President; Janet Hall, Secretary.

The Mathematics Club, Cornell College, Mt. Vernon, Iowa.

The regular meeting time is the third Thursday of every month. The permanent arrangements committee consists of Hazel Cory, Myrtle McIntosh, and Arnold Herkleman.

The programs for the year 1928–29 were as follows:
Nov. 1928. Dinner followed by an address by Prof. E. W. Chittenden. *Entertainment Committee*: Ethel Cain and Iva Shaffer.
Dec. Christmas party with recreational mathematics. *Committee*: Viola Smith, James Nauman, Irma Kaufman, Leona Barnes, and Albert Nelson. *Entertainment Committee*: Mary Schmeiser, Newell Lumsden, and Velda McCauley.
January, 1929. "The Calendar," by Professor E. E. Moots. *Entertainment Committee*: Pauline Davidson, Margaret Kopf, and Ernest Nielson.
February. (1) A Drama, "The evolution of numbers;" (2) Arnold Herkleman, "The history of $\sqrt{(-1)}$. *Committee*: Lilliam Frink, Beatrice Burge, Philip Switzer, Arthur Rouse. *Entertainment Committee*: Ruth Stewart, Grace Tielkemeir, William Shaw.
March. "The relation of mathematics to biology," by Professor H. M. Kelley. *Entertainment Committee*: Bernadine Burge, Florence Wentzel, Portia Tracy.

April. "Fundamental concepts," by Professor F. M. McGaw. *Entertainment Committee*: Ruth Ketzle, Gertrude Sayer, Elmer Meyers.
May. "Recent astronomical discoveries," by Leonard Hute; "Comets," by Albert Nelson. *Entertainment Committee*: Eleanor Martin, Gladys Berry, Evelyn McMeans.

There was also an afternoon meeting at which Professor Wiley, of Iowa City, spoke; and there was a picnic at the end of the year.

(Report by James L. Nauman)

The Delta Chi Fraternity of the University of New Hampshire, Durham, New Hampshire.

The officers for the year 1928–29 were: Real Dis Rochers, President; Louise Woodman, Vice-president; Florence M. Brown, Secretary; Philip Nudd, Treasurer.

The programs for the year 1928–29 were as follows:
January 19, 1928. "Diophantine analysis," by Jessie Daniels, '28.
February 9. Initiation banquet.
February 23. "The theory and uses of the planimeter," by Carroll Avery, '28.
March 8. "How trigonometric and logarithmic tables are made," by Malcolm Sargent, '28.
March 22. "Comparison of circular and hyperbolic functions," by Charles Morreels, '28.
April 5. "Partial differentiation theory and certain uses," by Priscilla Morris, '28.
April 19. "Mean value and probability theory and illustrations," by Lawrence Smith, '28.
May 3. Annual party.
May 17. Election of officers.
October 11. Election of committees.
October 25. "Hyperbolic functions," by Philip Nudd, '30.
November 8. Social.
November 22. "Vector analysis," by Marvin P. Salt.
January 17, 1929. "Mathematics and thermo-dynamics," by Assistant Professor Edward Donovan.
January 31. Annual Business Meeting.
February 14. Initiation Banquet.
February 28. "Mathematics and mechanics," by Assistant Professor Edward L. Getchell.
March 14. "Mathematics and electricity," by Mr. William B. Nulsen.

The Mathematics Club of George Washington University, Washington, D. C.

The officers for the year 1928–1929 were: Dr. F. E. Johnston, President; Mr. Michael Goldberg, Secretary.

The program for the year 1928–1929 was as follows:
October 15, 1928. "The calculus of tensors," by Professor Edgar W. Woolard.
October 29. "The complex quantity slide rule," by Mr. Michael Goldberg.
November 12. "The problem of Apollonius," by Dr. Paul Wernicke.
November 26. "Line values of the trigonometric functions and their use in constructing curves," by Dr. F. E. Johnston.
December 10. "Solution of equations by Graeffe's method," by Professor W. J. Berry.
January 7, 1929. "The story of numbers," by Dr. Tobias Dantzig.
February 18. "Short methods in arithmetic and algebra," by Mr. B. Z. McLeroy.
March 5. "Multiple points," by Mr. P. J. Federico.
March 18. "The three problems of antiquity," by Mr. D. B. Lloyd.
April 15. "Heaviside's operational calculus," by Dr. Louis Cohen.
April 29. "Special methods of integration," by Mr. S. J. Snyder.
June 8. Picnic at Chapel Point, Md.

(Report by Michael Goldberg)

The Mathematics Section of the Hanover College Science Club, Hanover, Indiana.

The officers for 1928–29 were: George N. Bishop, President; William Willis, Vice-President; Louisa Plummer, Secretary-Treasurer; Dr. Harvey A. Zinszer, Faculty Adviser.

The program for the year 1928–29 was as follows:

October 10, 1928. "History and evolution of the electric spark," by Dr. H. A. Zinszer; "Horner's synthetic division," by Mary E. Holderman, '29.

November 27. "Science and student life at Cambridge," by Dr. Mason Hufford, (Indiana University).

January 9, 1929. "Evolution of pi," by Jessie Hope Rankin, '29; A reel on the "Development of the telephone."

February 13. "Variation of volume with depth in horizontal tanks, by Thirza Kurtz, '29.

March 13. "Sturm's functions," by Cecil Collins, '29; "Curve tracing," by George Bishop, '29.

April 24. "Prime numbers," by Louisa Plummer, '29; A reel on "Revelations of the X-ray."

May 22. Picnic at Cedar Bluff; Election of officers; A reel on "Theory and operation of the radio."

The new officers for 1929–30 are: William Willis, President; Harry Francke, Vice-president; Helen Campbell, Secretary-Treasurer.

Three members of the club attended the state mathematical meeting at Culver Military Academy, where Margaret Darragh, '29, gave a paper on "Three methods for solving skew lines." Eight senior mathematics majors made Delta Epsilon, a national scientific honor fraternity.

(Report by Helen Campbell)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3394. *Proposed by Paul Wernicke, Washington, D. C.*

In space of three or more dimensions two n-gons A, B have the consecutive vertices $A_1, A_3, \cdots, A_{2n-1}$ and $B_2, B_4, \cdots, B_{2n}$. The feet of perpendiculars from A_1 to $B_{2n}B_2, \cdots$, from A_{2k-1} to $B_{2k-2}B_{2k}$ are $B_1 \cdots B_{2k-1}$. Those of perpendiculars from B_{2k} to $A_{2k-1}\,A_{2k+1}$ are A_{2k}. Subscripts are to be taken modulo $2n$, and $k=1, 2 \cdots n$.

(a) Show that

$$\sum_k (A_{2k-1}A_{2k})^2 - \sum (A_{2k}A_{2k+1})^2 = -\sum (B_{2k-2}B_{2k-1})^2 + \sum (B_{2k-1}B_{2k})^2$$

(b) What does it mean geometrically if the members of this equation both vanish?

3395. *Proposed by J. Rosenbaum, Milford, Conn.*

Given an n-gon, $A_1A_2 \cdots A_n$, show how to locate a point X such that the vectors XA_1, XA_2, $\cdots$, XA_n form a closed n-gon.

3396. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let D, E, F be the feet of the altitudes, and A', B', C', the mid-points of the sides of a triangle ABC. Show that the double elements of the three involutions (BC, DA'), (CA, EB'), (AB, FC') are three pairs of opposite vertices of a complete quadrilateral.

3397. *Proposed by V. Rojansky, Washington University.*

Let $L_k(u) = e^u(d^k/du^k)(u^ke^{-u})$ be the Laguerre polynomial of the kth order. Let $L_k^m(u)$ be the mth derivative with respect to u of the Laguerre polynomial of the kth order.

Let n, l, q, and p be integers such that $n>0$, $q>0$, $0 \leqq l \leqq n-1$, and $0 \leqq p \leqq q-1$.

Then show that the algebraic sign of the definite integral

$$\int_0^\infty x^{l+p+3}e^{\mu}L_{n+l}^{2l+1}\left(\frac{x}{n}\right)L_{q+p}^{2p+1}\left(\frac{x}{q}\right)dx, \quad \text{where} \quad \mu = -\tfrac{1}{2}(n^{-1}+q^{-1})x,$$

is the same for $p=l+1$ as it is for $p=l-1$, when either value of p is permitted by the conditions given above.

The proposer infers the truth of this proposition from the physical considerations involved in the theory given in the Physical Review, vol. 33 (1929), p. 1. It is of interest to prove this on purely mathematical grounds. Integrals of this type were discussed by Schroedinger, Annalen der Physik, vol. 80 (1926), p. 437. It may be added that the Laguerre polynomials are closely related to Whittaker's W function. (See Whittaker and Watson's *Modern Analysis*.)

SOLUTIONS

3174 [3170; 1926, 104]. *Proposed by R. H. Sciobereti, University of California.*

Given the base BC of a spherical triangle in position and magnitude and given the magnitude of the angle A which is opposite to BC, find the locus of A.

Solution by Otto Dunkel, Washington University.

Let M be the middle point of the base, the extremities of which will be denoted by B, B', $B'M=MB=b$; and let the vertex P be located so that angle $B'PB=A$. Let also a set of rectangular axes be taken with the origin at O, the center of the sphere, so that OM is along the positive z-axis and the plane of OBB' is the yz-plane. Then the direction cosines of OB and OB' are respectively $(0, \sin b, \cos b)$, $(0, -\sin b, \cos b)$. Let α, β, γ be the direction angles of OP, and denote the angles $B'OP$ and BOP by p' and p. Then

$$\cos p = \cos\beta \sin b + \cos\gamma \cos b, \qquad \cos p' = -\cos\beta \sin b + \cos\gamma \cos b,$$

and from the triangle $B'BP$ we have

$$\cos 2b - \cos p \cos p' = \sin p \sin p' \cos A,$$

$$\cos 2b - \cos^2 \gamma \cos^2 b + \cos^2 \beta \sin^2 b = \sin p \sin p' \cos A.$$

If now we square both sides, replace $\sin^2 p$ and $\sin^2 p'$ by values obtained from the equations above, and then make some reductions, we obtain the equation of the cone passing through the desired locus,

$$(1)\quad [\cos^2 b(x^2 + y^2) - \sin^2 b(x^2 + z^2)]^2 \sin^2 A - x^2[x^2 + y^2 + z^2] \sin^2 2b \cos^2 A = 0.$$

If $A = 90°$ the equation reduces to one of the second degree (repeated),

$$(2)\qquad \cos^2 b(x^2 + y^2) - \sin^2 b(x^2 + z^2) = 0.$$

We have here the equation of a particular case of a spherical conic. We shall derive the focal property for such conics. Let F and F' be two points on the great circle of the sphere determined by the xz-plane at the distances c and $-c$ from M, and suppose that P is located on the sphere so that $FP + F'P = 2a$. The direction cosines of OF, OF' and OP are $(\sin c, 0, \cos c)$, $(-\sin c, 0, \cos c)$, $(\cos \alpha, \cos \beta, \cos \gamma)$ respectively. If we set $FP = \rho$, $F'P = \rho'$, we have at once

$$\cos \rho = \cos \alpha \sin c + \cos \gamma \cos c, \cos \rho' = -\cos \alpha \sin c + \cos \gamma \cos c.$$

Hence

$$2 \cos \gamma \cos c = \cos \rho + \cos \rho' = 2 \cos \tfrac{1}{2}(\rho + \rho') \cos \tfrac{1}{2}(\rho - \rho') = 2 \cos a \cos \tfrac{1}{2}(\rho - \rho'),$$

$$2 \cos \alpha \sin c = \cos \rho - \cos \rho' = -2 \sin \tfrac{1}{2}(\rho + \rho') \sin \tfrac{1}{2}(\rho - \rho')$$

$$= -2 \sin a \sin \tfrac{1}{2}(\rho - \rho').$$

By squaring and combining these two equations we obtain

$$\cos^2 \gamma \sin^2 a \cos^2 c + \cos^2 \alpha \cos^2 a \sin^2 c - \cos^2 a \sin^2 a = 0.$$

The constant term may be multiplied by $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ in order to make the equation homogeneous. It then becomes

$$(3)\quad \cos^2 a (\sin^2 a - \sin^2 c)x^2 + \sin^2 a \cos^2 a\, y^2 - \sin^2 a (\cos^2 c - \cos^2 a)z^2 = 0.$$

We shall now identify (3) with (2). If $b = 45°$ it is easily seen geometrically that the locus of P for angle $B'PB = 90°$ is two great circles passing through the pole of $B'B$ and its extremities. If $90° > b > 45°$, extend the arc BB' to B_1 so that $BB_1 = 180°$; also extend $B'B$ to B_1' so that $B'B_1' = 180°$. The arcs B_1B' and BB_1' are diametrically opposite, and the locus will be obtained by making B_1PB' and $BP'B_1'$ right angles. Hence we need to consider only the case where $b < 45°$. The locus of P, angle $B'PB = 90°$, has intercepts on the meridian circle at M which may be denoted by $a = MA$ and $-a = MA'$. In the right triangle AMB, angle $MAB = 45°$ and hence $a > b$ and $\tan b = \sin a$. Locate the points F and F' on AA' so that $FB = F'B = a$, and set $MF = c$; then $\cos a = \cos b \cos c$, and from the equation above $\sin b = \cos b \sin a$. Hence we derive $\sin^2 a \cos^2 a = \cos^2 c - \cos^2 a = \sin^2 a - \sin^2 c$. After inserting these values in (3)

that equation reduces to $x^2\cos^2 a+y^2-z^2\sin^2 a=0$, and by use of $\sin a=\tan b$ this last equation reduces to (2).

This shows that, if in a spherical ellipse with major axis $A'A$ and minor axis $B'B$ the angle BAB' is a right angle, then if P is any other point on the ellipse BPB' is also a right angle.

3348 [1928, 446]. *Proposed by A. C. Aitken, University of Edinburgh.*

Show that

$$\frac{\pi}{6}=\sum_{1}^{\infty}\operatorname{arccot} 2u_r^2,$$

where $u_r=4u_{r-1}-u_{r-2}$ with $u_1=1$, $u_2=3$.

Solution by Harry Langman, Arverne, L. I.

We have

$$\cot\frac{\pi}{6}=\sqrt{3}=1+\frac{1}{1+}\,\frac{1}{2+\cdots}, \tag{1}$$

the partial quotients 1, 2, repeating. Let p_t/q_t be the tth convergent. Then

$$p_{2r+1}=p_{2r+2}-p_{2r},\quad p_{2r+3}=2p_{2r+2}+p_{2r+1};$$

whence,

$$p_{2r+4}=p_{2r+3}+p_{2r+2}=4p_{2r+2}-p_{2r}, \tag{2}$$

a recurring relation for the even convergents by themselves. It also follows that

$$q_{2r+4}=4q_{2r+2}-q_{2r}. \tag{3}$$

We now set

$$a_r=\operatorname{arccot}\frac{p_{2r+2}}{q_{2r+2}}-\operatorname{arccot}\frac{p_{2r}}{q_{2r}}. \tag{4}$$

Here $a_r>0$. We may write

$$\frac{\pi}{6}=\operatorname{arccot}\frac{p_{2t}}{q_{2t}}+\sum_{r=t}^{\infty}a_r. \tag{5}$$

From (4),

$$\cot a_r=\frac{p_{2r+2}p_{2r}+q_{2r+2}q_{2r}}{p_{2r}q_{2r+2}-q_{2r}p_{2r+2}}.$$

By the use of (2) and (3), the denominator reduces to $p_{2r-2}q_{2r}-q_{2r-2}p_{2r}$, hence to $p_2q_4-q_2p_4$, or 1. Hence,

$$\cot a_r=p_{2r+2}p_{2r}+q_{2r+2}q_{2r}. \tag{6}$$

We now note that

$$p_{2r}=2p_{2r-2}+3q_{2r-2}\quad\text{and}\quad p_{2r-2}=2p_{2r-4}+3q_{2r-4} \tag{7}$$

hold for the initial values of r. Hence, by (2) and (3), $p_{2r+2}=2p_{2r}+3q_{2r}$, and (7) holds universally. In similar fashion we find that

$$q_{2r} = p_{2r-2} + 2q_{2r-2}. \tag{8}$$

From (6), (7), and (8), we have

$$\cot a_r = (2p_{2r} + 3q_{2r})p_{2r} + (p_{2r} + 2q_{2r})q_{2r} = 2(p_{2r} + q_{2r})^2. \tag{9}$$

Hence, (5) becomes

$$\frac{\pi}{6} = \operatorname{arccot}\frac{p_{2t}}{q_{2t}} + \sum_{r=t}^{\infty} \operatorname{arccot} 2(p_{2r} + q_{2r})^2. \tag{10}$$

If we set $u_{r+1}=p_{2r}+q_{2r}$, and choose $t=1$, (10) becomes

$$\frac{\pi}{6} = \operatorname{arccot} 2 + \sum_{r=2}^{\infty} \operatorname{arccot} 2u_r^2 = \sum_{r=1}^{\infty} \operatorname{arccot} 2u_r^2. \tag{11}$$

From (2) and (3) we obtain the recurrence relation $u_{r+1}=4u_r-u_{r-1}$.

If now we consider the sequence of odd convergents in (1), and set

$$b_r = \operatorname{arccot}\frac{p_{2r-1}}{q_{2r-1}} - \operatorname{arccot}\frac{p_{2r+1}}{q_{2r+1}}, \tag{4'}$$

then $b_r > 0$, and

$$\frac{\pi}{6} = \operatorname{arccot}\frac{p_{2t-1}}{q_{2t-1}} - \sum_{r=t}^{\infty} b_r. \tag{5'}$$

We find $\cot b_r = (p_{2r-1}+q_{2r-1})^2$, whence

$$\frac{\pi}{6} = \operatorname{arccot}\frac{p_{2t-1}}{q_{2t-1}} - \sum_{r=t}^{\infty} \operatorname{arccot}(p_{2r-1} + q_{2r-1})^2. \tag{10'}$$

If we set $v_r=p_{2r-1}+q_{2r-1}$ and choose $t=1$,

$$\frac{\pi}{6} = \frac{\pi}{4} - \sum_{r=1}^{\infty} \operatorname{arccot} v_r^2, \tag{11'}$$

with $v_{r+1}=4v_r-v_{r-1}$ and initial values $v_1=2$, $v_2=8$.

Note by the Editors. This problem is similar to problem 3051 [1924, 49] proposed by Norman Anning. A solution [1925, 386] was given by the proposer which, after a slight modification, shows that

$$\frac{\pi}{12} = \operatorname{arccot}\frac{u_2}{u_1} + \sum_{r=1}^{\infty} \operatorname{arccot}\left(\frac{4u_{r+1}^2}{D}\right),$$

where $u_{r+2}=4u_{r+1}-u_r$ and $D=u_2^2-u_1u_3$. This result assumes different forms according to the values assigned to u_1 and u_2. If $u_1=2$, $u_2=8$, then $D=4$ and we have the result given by Anning, which is also the second result given above.

If $u_1=1$, $u_2=1$, $u_3=3$, then $D=-2$ and we obtain the result in this problem after a slight modification.

This method may be modified by using other suitable difference equations. Thus the equation $x^2-2x-1=0$ has the root $1+\sqrt{2}=\cot(\pi/8)$, and this root is equal to Limit u_{r+1}/u_r $(r\to\infty)$, where $u_{r+2}=2u_{r+1}+u_r$. Hence by a similar analysis

$$\frac{\pi}{8}=\operatorname{arccot}\frac{u_2}{u_1}+\sum_{r=1}^{\infty}(-1)^{r+1}\operatorname{arccot}\left[\frac{2u_{r+1}(u_{r+1}+u_r)}{D}\right].$$

Thus, if $u_1=1$, $u_2=1$, then $D=-2$ and we find that

$$\frac{\pi}{8}=\sum_{r=1}^{\infty}(-1)^{r+1}\operatorname{arccot}\left[u_{r+1}(u_{r+1}+u_r)\right].$$

3331 [1928, 321]. *Proposed by Otto Dunkel, Washington University.*

In a spherical triangle ABC such that $AC<BC$ two points A', B', are taken on the side AB so that $\angle A'CA=\angle BCB'\leqq C/2$. Prove that AA' is greater than, equal to, or less than BB' according as $AC+CB$ is greater than, equal to, or less than 180°.

Solution by the Proposer.

Consider first the case $AC+CB<180°$. Since $BC>AC$ it follows that $AC<90°$ and $A>B$. Extend the arcs CA and CB until they meet again at C', and let L and N be the mid-points of CAC' and CBC', respectively. Then the arc LN is less than 180° since $C<180°$. Draw the arc CM_1C' bisecting LN at M_1 and cutting AB at M, and then produce AM_1 to meet CBC' in B_1. The two right triangles AM_1L and B_1M_1N are congruent, and hence $AC+CB_1=180°$, and also $AM_1=M_1B_1<90°$. Thus B lies within the segment B_1C. Since the triangle ACB lies within ACB_1 we have $A+B<180°$; thus $B<A$, $B<180°-A$, $B<90°$.

Now let CA' and CB' cut AB_1 in A'' and B'', then $CA'+CB'<CA''+CB''=180°$. Lay off on CB the arc $CA_1=CA$, and draw the arc A_1M cutting CB' in A_1'. Then A_1 falls within CN, $CA_1'=CA'<90°$, and $A_1A'_1=A'A$. Since $CA_1'+CB'=CA'+CB'<180°$, the mid-point of $A_1'B'$ lies between the arc LN and the point C, and hence B' is nearer LN than A_1'. If then $B'N_b$ and $A_1'N_a$ are the perpendiculars to CNC', $N_aA_1'<N_bB'$, since $\angle BCB'<90°$. The point N_a lies within NC as well as A_1, and hence $N_aA_1<90°$, also $N_aA_1'<NM_1<90°$; and it follows that $A_1A_1'<90°$. Now if $BB'\geqq 90°$, then $BB'>A_1A_1'=A'A$ and the theorem is true. Hence we need to consider only the case $BB'<90°$. In the right triangle N_bBB', $BB'<90°$, and $\angle N_bBB'=B<90°$; hence N_bB' and N_bB are each less than 90°. Lay off on the prolongation of BN_b the arc $N_bK=N_aA_1$ (in absolute value) and on N_bB', the arc $N_bH=N_aA_1'$. Draw the arcs KH and KB'. Then $\angle N_bKH=\angle N_aA_1A_1'$. If $A\leqq 90°$, $\angle N_aA_1A_1'=A$; if $A>90°$, $\angle N_aA_1A_1'=180°-A$. But in either case it follows that $B<\angle N_bKH$. Since $N_bH<N_bB'$ and these two arcs as well as N_bK are each less than 90°,

$B < \angle N_b KH < \angle N_b KB'$. Since $BK < 180°$, it now follows that $BB' > KB' > KH$, or $BB' > A'A$, and the proof is complete.

If $AC + CB = 180°$, then $A' \equiv A''$, $B' \equiv B''$, $B \equiv B_1$, and $BB' = A'A$. If $AC + CB > 180°$ and $BC > AC$, then $AC' + C'B < 180°$ and $AC' > BC'$. Hence from the above proof $AA' > B'B$.

Note. In spherical geometry the condition $AC + CB < 180°$ insures the fact that the external angle at A of a triangle ABC is greater than the internal angle at B. In the geometries for which the sum of the angles of a triangle is less than or equal to 180° this inequality of angles is always true and the proof that $BB' > A'A$ is simpler. We draw as before CM bisecting the angle BCA and take on CB the length $CA_1 = CA$. Then draw A_1M cutting CB' in A_1'. The corresponding parts of the triangles CA_1A_1' and CAA' are equal, and $CB' > CA_1'$. Then if $B'N_b$ and $A_1'N_a$ are the perpendiculars to CB, $N_aA_1' < N_bB'$. Also $B < A$ and $B < 180° - A$, and the proof follows as before.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. William E. Wickenden has assumed his work as president of the Case School of Applied Science. Fitting ceremonies commemorating the founding of the School in 1880 will be held sometime during the year, possibly in conjunction with the installation of Dr. Wickenden as third president of the School within the fifty year period.

Professor W. L. Crum, of Stanford University, has been appointed professor of economics at Harvard University.

Dr. H. B. Curry has been appointed assistant professor of mathematics at the Pennsylvania State College.

Assistant Professor C. C. MacDuffee, of the Ohio State University, has been promoted to an associate professorship.

Mr. E. D. McCarthy, of the Pennsylvania State College, has accepted a position at the University of Detroit, and Miss Frances Monteith of the same college has gone into statistical work.

Assistant Professor Florence M. Mears, of the Pennsylvania State College, has been appointed assistant professor at George Washington University.

Mr. Albert E. Meder, Jr. has been promoted to an assistant professorship at the New Jersey College for Women.

Assistant Professor Gaylord M. Merriman, of Grinnell College, has been appointed assistant professor of mathematics at the University of Cincinnati.

Dr. John A. Miller, of Swarthmore College, is retiring from teaching and administrative work to become research professor of astronomy. He will continue in charge of the Sproul Observatory.

Dr. Hillel Poritsky has accepted a position with the General Electric Company.

Professor O. H. Rechard, head of the Department of Mathematics in the University of Wyoming, is on leave of absence in the University of Wisconsin. Assistant Professor C. F. Barr is acting head of the Department.

Professor J. Shibli and Mr. W. O. Rogers, of the Pennsylvania State College are on leave of absence; Professor Shibli is studying at Teachers College, Columbia University, and Mr. Rogers, at the University of Chicago.

Professor Charles C. Wagner and Mr. O. J. Farrell, of the Pennsylvania State College remain on leave of absence; Professor Wagner will study at the University of Michigan, and Mr. Farrell at Harvard University.

Professor Norbert Weiner, of the Massachusetts Institute of Technology, is visiting professor of physics at Brown University.

Dr. Harvey A. Zinszer who for the past two years has been professor of physics and acting professor of mathematics at Hanover College has been appointed professor of physics and astronomy at the Kansas State Teachers' College.

The following appointments to instructorships are announced:

New Jersey College, Mr. Robert M. Walter.
Rutgers University, Mr. Hubert B. Huntly.
Pennsylvania State College, Mr. C. H. Graves,
Mr. William Mann, Miss Gladys Quigg,
Dr. Leo Zippin.
South Dakota State School of Mines, Mr. Harry Pool.
University of Wyoming, Mr. Richard W. Warner.

The Fourth Carus Mathematical Monograph

The Carus Monograph Committee is pleased to announce that the fourth number is now in process of publication and will be ready for distribution by the time of the annual meetings in Des Moines. The title of this Monograph is "Projective Geometry" by Professor JOHN W. YOUNG of Dartmouth College, now President of the Association. The preceding numbers are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS, (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ.

The price of these Monographs is $1.25 to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 339 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

As heretofore, for the convenience of members, the forthcoming Monograph will be charged along with the bill for annual dues late in December. (This item may be cancelled in case it is not wanted.) New members and those who have neglected to subscribe for the previous numbers may still do so by ordering directly from the Secretary. As the series goes on, the complete list of Monographs will become more valuable if not indispensable to the individual library of an increasing number of members as well as to most college and all university libraries. It is gratifying to announce that the sales of the preceding numbers are continuing very favorably and that two of them have already gone to second editions. It would be still more gratifying if a larger proportion of members (now somewhat more than fifty per cent) should become regular subscribers to this Monograph series. Failure to do so is, doubtless, in many cases due to oversight or procrastination. Now is a good time to remedy such a condition.

Attention is called to the enlargement of the membership of the Monograph Committee by the addition of Professors AUBREY J. KEMPNER of the University of Colorado, and JOHN W. YOUNG, of Dartmouth College. The other members of the Committee are: Professor GILBERT AMES BLISS, of the University of Chicago; Professor DAVID RAYMOND CURTISS, of Northwestern University; and Professor HERBERT ELLSWORTH SLAUGHT, of the University of Chicago.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1929.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA, Fairfield, Iowa, April 26-27.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY, Lexington, Ky., April 13.
LOUISIANA-MISSISSIPPI, Lafayette, La., April 12-13.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, Minn., May 11.
MISSOURI, Kansas City, Mo., November 16
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 19-20.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

DIFFERENTIAL EQUATIONS AS A FOUNDATION FOR ELECTRICAL CIRCUIT THEORY

By THORNTON C. FRY, Bell Telephone Laboratories

1. *Introductory Remarks.* The subject of electrical circuit theory is of fundamental importance, not only in the field of communication, but in power engineering as well. From the physical standpoint it is naturally a study in real quantities, since the physical forces and currents are themselves real, and the mathematical methods originally used in connection with it dealt almost exclusively with real variables. During the present century, however, a progressive transition has taken place, until now the use of complex quantities is by all odds the usual thing, and real numbers are rather rare. This transition has been brought about, not through the influence of mathematicians who have taught the engineers the advantage of complex quantities, but rather through the initiative of a few engineers who have preached the idea to their own kind. As a natural consequence the duty of explaining why *imaginary* numbers are so peculiarly appropriate in a problem in which the physical quantities are so obviously *real* has fallen to the professor of electrical engineering, though the reasons are essentially mathematical in character, and can be taught much more concisely and effectively in connection with a study of elementary differential equations. It is my purpose to emphasize this fact by outlining as briefly as possible the nature of the problem, and the ideas upon which its solution is based.

Perhaps the matter would be of little consequence if the results were satisfactory, but this is not the case. For, simple as we shall find the ideas to be when presented in a suitable setting, they are really so foreign to the subject of electricity itself that it is not easy to explain them effectively in a technical course. Hence, even if the instructor has the proper perspective, which is not always the case, he usually meets his immediate needs by some sort of vectorial analogy, and the student is left with a very hazy idea as to what it is all about. The accuracy of this statement will be readily attested by those mathematicians who have occasion to meet representative groups of such students after graduation.

2. *The Differential Equations of Circuit Theory.* The differential equations of elementary circuit theory—and the same is true of the simpler dynamical systems in general—are characterized by five principal characteristics:

1. They are linear.

Of course, any dynamical system can be overloaded to the point where it becomes non-linear. Very frequently, however, its practical usefulness ceases when non-linearity sets in, and because of this fact we are justified in thinking of it as a linear system. There is also another class of dynamical systems, of which vacuum-tube devices are excellent electrical illustrations, many of whose uses depend upon the fact that they are essentially non-linear in character. Such systems, however, require special methods of treatment, and are not usually regarded as part of the subject-matter of "elementary" circuit theory. More-